Dynamic Programming with State-Dependent Discounting[1]

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Abstract. This paper extends the core results of discrete time infinite horizon dynamic programming to the case of state-dependent discounting. We obtain a condition on the discount factor process under which all of the standard optimality results can be recovered. We also show that the condition cannot be significantly weakened. Our framework is general enough to handle complications such as recursive preferences and unbounded rewards. Economic and financial applications are discussed.

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1. Introduction

Researchers in economics and finance routinely adopt settings where the subjective discount rate used by agents in their models varies with the state. For example, Albuquerque et al. (2016) study an asset pricing model in which the discount rate is perturbed by an AR(1) process. They show that the resulting demand shocks help explain the equity premium puzzle. Mehra and Sah (2002) find that small fluctuations in agents' discount factors can have large effects on equity price volatility. Schorfheide et al. (2018) and Gomez-Cram and Yaron (2020) likewise embed state-dependent discount factors into Epstein–Zin preferences to generate realistic asset prices and returns.

State-dependent and time-varying discount rates are also common in studies of savings, income and wealth. An early example is Krusell and Smith (1998). In related work, Krusell et al. (2009) model the discount process as a three state Markov chain and show how discount factor dispersion helps their heterogeneous agent model match

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the wealth distribution. Fagereng et al. (2019) use time-varying discount rates and portfolio adjustment frictions to explain the positive correlation between savings rates and wealth observed in Norwegian panel data. Hubmer, Krusell, and Smith (2020) model discount dynamics using a discretized AR(1) process.

State-dependent discounting is also found in analyses of fiscal and monetary policy. For example, Eggertsson and Woodford (2003) study monetary policy in the presence of zero lower bound restrictions with dynamic time preference shocks. Woodford (2011) considers the government expenditure multiplier in a similar environment. Eggertsson (2011) and Christiano, Eichenbaum, and Rebelo (2011) study the effect of fiscal policies at the zero lower bound on interest rates, while Nakata and Tanaka (2020) analyze the term structure of interest rates at the zero lower bound when agents have recursive preferences. In all of these models, state-dependent variation in discount rates plays a significant role.[2]

In addition, state-dependent discounting is often used in studies of macroeconomic volatility. For example, Primiceri et al. (2006) argue that shocks to agents' rates of intertemporal substitution are a key source of macroeconomic fluctuations. Justiniano and Primiceri (2008) study the shifts in the volatility of macroeconomic variables in the US and find that a large portion of consumption volatility can be attributable to the variance in discount factors. Additional research in a similar vein can be found in Justiniano et al. (2010), Justiniano et al. (2011), Christiano et al. (2014), Saijo (2017), and Bhandari et al. (2013).

The standard theory of dynamic programming over infinite horizons (see, e.g., Blackwell (1965), Stokey et al. (1989), or Bertsekas (2017)) does not accommodate state-dependent discounting. Instead, it assumes either zero discounting (and considers long-run average optimality) or a constant and positive discount rate, which corresponds to a discount factor strictly less than one. This implies that, in the canonical setting, the Bellman operator satisfies the conditions of Banach's contraction mapping theorem, which in turn provides the foundations for the standard optimality theory.

We reconsider the standard theory when the constant discount factor β is replaced by a discount process $\{\beta_t\}$, so that time t payoff π_t is discounted to present value as $\mathbb{E}_z \prod_{i=0}^{t-1} \beta_i \, \pi_t$ rather than $\beta^t \, \mathbb{E}_z \, \pi_t$. Here z is the initial condition of an exogenous Markov state process that drives evolution of the discount factor. We replace the traditional condition $\beta < 1$ with a weaker "eventual discounting" condition: existence of a $t \in \mathbb{N}$ such that $\sup_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{i=0}^{t-1} \beta_i < 1$. For a finite irreducible state process,

[2]See also Correia et al. (2013), Hills and Nakata (2018), Hills et al. (2019) and Williamson (2019).

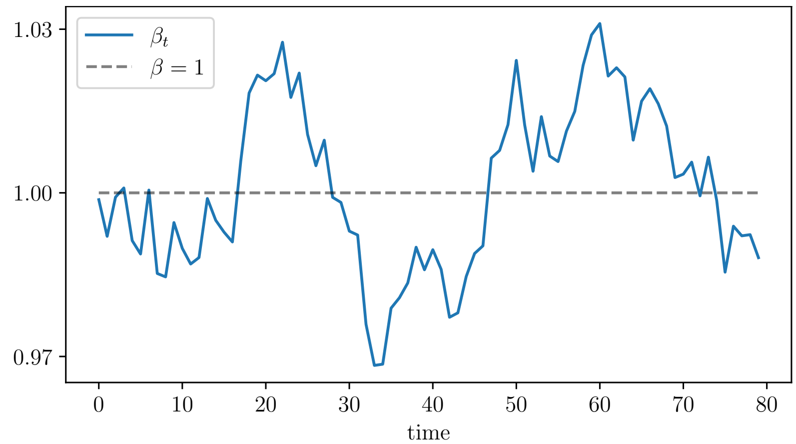

FIGURE 1. Simulated time path for $\{\beta_t\}$ in Hills et al. (2019)

this is equivalent to existence of a $t \in \mathbb{N}$ such that $\mathbb{E} \prod_{i=0}^{t-1} \beta_i < 1$, where $\mathbb{E}$ is the unconditional expectation.[3]

We show that, when eventual discounting holds, (i) the value function satisfies the Bellman equation, (ii) an optimal policy exists, (iii) Bellman's principle of optimality holds, and (iv) value function iteration and Howard policy iteration (Howard, 1960) are both convergent. When β_t is constant at $\beta < 1$, eventual discounting holds at $t = 1$, so these results capture the standard theory as a special case.

Our conditions do not rule out $\beta_t \geqslant 1$ with positive probability. One example of why this matters is provided by the New Keynesian literature, where the discount factor is often allowed to temporarily attain or exceed unity, so that the zero lower bound on the nominal interest rates binds. For example, Christiano et al. (2011) admit a shock where $\beta = 1.02$ in their study of the government spending multiplier. Similarly, Hills et al. (2019) analyze tail risk associated with the effective lower bound on the policy rate in a model where the discount process is a constant multiple of a discretized AR(1) process that regularly generates value of β_t exceeding unity. Figure 1 illustrates by showing a simulated time path of $\{\beta_t\}$ using their parameters.[4]

[3]As stated above, we assume that the discount factor is driven by an exogenous state process. However, our methods can also be applied to individual agent problems where endogenous aggregates appear in the discount factor, provided that the agent treats these aggregates as external to his or her actions. See, for example, Schmitt-Grohé and Uribe (2003).

[4]The specification is based around an AR(1) process and detailed in Example 4.3 below. Other studies using an AR(1) specification for the discount process or its logarithm include Nakata (2016), Hubmer et al. (2020), Albuquerque et al. (2016) and Schorfheide et al. (2018).

We discuss the eventual discounting condition at length in the paper, giving several equivalent conditions. One of these involves a bound on the spectral radius of a discounting operator. This connects our work to a strand of literature in finance that studies the long-term factorization of stochastic discount factors using eigenfunctions of valuation operators (see, e.g., Hansen and Scheinkman (2009), Hansen and Scheinkman (2012), and Qin and Linetsky (2017)). Drawing on these ideas, Borovička and Stachurski (2020) and Christensen (2020) connect the spectral radius of valuation operators with existence and uniqueness of recursively defined utilities. However, neither of these papers provides results on optimality or dynamic programming.

To handle unbounded rewards, we extend two approaches that have been developed previously for the case of constant discounting. The first one treats homogeneous programs in the spirit of Alvarez and Stokey (1998) and Stokey et al. (1989, Section 9.3). The second uses a local contraction method pioneered in Rincón-Zapatero and Rodríguez-Palmero (2003) and further developed by Martins-da Rocha and Vailakis (2010) and Matkowski and Nowak (2011). In each case, we show how the eventual discounting condition can be adapted to handle these extensions.

In addition, we study dynamic programming with Epstein-Zin utilities, where rewards are unbounded above and the Bellman operator is not a contraction in the short or long run under standard metrics. To solve the problem, we extend earlier work by Marinacci and Montrucchio (2010), Bloise and Vailakis (2018), and Becker and Rincón-Zapatero (2018), which exploits the monotonicity and concavity of the aggregator, to allow for state-dependent discounting. We show that, in the case of Epstein–Zin utility, the eventual discounting condition must be adapted to compensate for the role played by elasticity of intertemporal substitution.

Other papers have analyzed dynamic programming problems where discount rates can vary. For example, Karni and Zilcha (2000) study the saving behavior of agents with random discount factors in a steady-state competitive equilibrium. Cao (2020) proves the existence of sequential and recursive competitive equilibria in incomplete markets with aggregate shocks in which agents also have state-dependent discount factors. In the mathematical literature, various issues in dynamic programming with state-dependent discounting have been studied; see, for example, Jasso-Fuentes et al. (2020) and the references therein.[5] However, these papers assume that the discount process

[5]Jasso-Fuentes et al. (2020) also allow the discount process to be endogenous, a case not covered in our framework. In economic applications, this often comes in the form of Uzawa type preferences (Uzawa, 1968) that are common in open economy models where discount factors are dependent on consumption. See Uribe and Schmitt-Grohé (2017) for a review. However, these models can be

in the dynamic program is bounded above by one or by some constant less than one.[6] This is too strict for many applications, as discussed above.

Our work is related to Toda (2019), who investigates an income fluctuation problem in which the agent has CRRA utility. He obtains a necessary and sufficient condition for the existence of a solution to the optimal saving problem with state-dependent discount factors. Ma et al. (2020) relax the CRRA restriction by constructing optimality results via a consumption policy operator. Their results are specialized to optimal savings with additively separable rewards and do not apply to problems that involve discrete choices, endogenous labor supply, durable goods, or other common features. In contrast, the theory below is developed in a general dynamic programming setting, where the state spaces are arbitrary metric spaces.

In addition, the consumption policy operator, around which the theory in Toda (2019) and Ma et al. (2020) is constructed, is defined from the Euler equation, which characterizes the solution in their setting. However, many recent applications of state dependent discounting use recursive preferences (see, e.g., Albuquerque et al. (2016), Basu and Bundick (2017), Schorfheide et al. (2018), Nakata and Tanaka (2020), or de Groot et al. (2020)), implying that the Euler equation contains the value function and the consumption policy operator methods break down. Our theory extends to recursive preferences and illuminates the role of elasticity of intertemporal substitution on eventual discounting.

The rest of this paper is structured as follows. Section 2 sets out the model and provides our main results. Section 3 gives applications. Section 4 reviews our key assumption. Sections 5 and 6 treat extensions. Section 7 concludes.

2. A Dynamic Program

In what follows, for any metric space Y, the symbols $m\mathsf{Y}$, $bm\mathsf{Y}$ and $bc\mathsf{Y}$ denote the (Borel) measurable, bounded measurable and bounded continuous functions from Y to $\mathbb{R}$ respectively. Unless otherwise stated, the last two spaces are endowed with the supremum norm and this norm is represented by $\| \cdot \|$. In expressions with products below, we adopt the convention that $\prod_{t=0}^{n-1} \beta_t = 1$ whenever $n = 0$.

treated using traditional dynamic programming techniques, since the discount factors are assumed to be strictly less than one in the literature.

[6] Schäl (1975) admits state-dependent discounting in discrete time under weaker conditions, but he directly assumes that expected discounted rewards are finite under any Markov policy. This restricts all primitives in the dynamic program simultaneously and makes the condition impractical for applications.

2.1. **Framework.** The state of the world consists of a pair (x, z), where x and z represent endogenous and exogenous variables. These variables take values in separable metric spaces X and Z respectively. The agent responds to (x, z) by choosing future state x' from $\Gamma(x, z) \subset \mathsf{X}$, where Γ is the *feasible correspondence*. Let $\operatorname{gr}\Gamma$ be the graph of Γ, defined by

$$\operatorname{gr}\Gamma = \{(x, z, x') \in \mathsf{S} \times \mathsf{X} : x' \in \Gamma(x, z)\} \quad \text{where } \mathsf{S} := \mathsf{X} \times \mathsf{Z}. \tag{1}$$

Similar to Bertsekas (2013), we combine the remaining elements of the dynamic programming problem into a single *continuation aggregator* H, with the understanding that $H(x, z, x', v)$ is the maximal value that can be obtained from the present time under the continuation value function v, given current state (x, z) and next period state x'. The aggregator H maps each (x, z, x', v) in $\operatorname{gr}\Gamma \times bm\mathsf{S}$ into $\mathbb{R}$ and is assumed to satisfy, for all $v, w \in bm\mathsf{S}$ and all $(x, z, x') \in \operatorname{gr}\Gamma$,

$$H(x, z, x', v) \leqslant H(x, z, x', w) \text{ whenever } v \leqslant w. \tag{2}$$

This basic monotonicity condition is satisfied in all applications of interest. Bellman's equation takes the form

$$v(x, z) = \sup_{x' \in \Gamma(x,z)} H(x, z, x', v). \tag{3}$$

For fixed X and Z, a *dynamic program* $\mathcal{D} = (\Gamma, H)$ consists of a feasible correspondence Γ and a continuation aggregator H.

2.2. **Feasibility and Optimality.** Let $\mathcal{D} = (\Gamma, H)$ be a dynamic program and let Σ be the set of *feasible policies*, defined as all Borel measurable maps σ from S to X such that $\sigma(x, z) \in \Gamma(x, z)$ for each (x, z) in S. Given such σ, let T_σ be the *policy operator* on $bm\mathsf{S}$ given by

$$(T_\sigma v)(x, z) = H(x, z, \sigma(x, z), v). \tag{4}$$

Define the *Bellman operator* T on $bm\mathsf{S}$ by

$$(Tv)(x, z) = \sup_{x' \in \Gamma(x,z)} H(x, z, x', v). \tag{5}$$

Given v_0 in $bm\mathsf{S}$ and σ in Σ, we can interpret $v_{n,\sigma}(x, z) := (T_\sigma^n v_0)(x, z)$ as the lifetime payoff of an agent who starts at state (x, z), follows policy σ for n periods and uses v_0 to evaluate the terminal state. The *σ-value function* for an infinite-horizon problem is defined here as

$$v_\sigma(x, z) := \lim_{n\to\infty} v_{n,\sigma}(x, z). \tag{6}$$

The definition requires that this limit exists and is independent of v_0. Below we impose conditions such that this is always the case.

We define the *value function* corresponding to our dynamic program by

$$v^*(x,z) = \sup_{\sigma \in \Sigma} v_\sigma(x,z) \tag{7}$$

at each (x,z) in S. A policy $\sigma^* \in \Sigma$ is called *optimal* if it attains the supremum in (7) at each (x,z) in S. We say that *Bellman's principle of optimality holds* when

$$\sigma \in \Sigma \text{ is optimal} \quad \iff \quad \sigma(x,z) \in \operatorname*{arg\,max}_{x' \in \Gamma(x,z)} H(x,z,x',v^*) \text{ for each } (x,z) \text{ in } \mathsf{S}.$$

2.3. **Assumptions.** A dynamic program $\mathcal{D} = (\Gamma, H)$ will be called *regular* if

(a) Γ is continuous, nonempty, and compact valued and

(b) the function $(x,z,x') \mapsto H(x,z,x',v)$ is bounded and measurable on $\operatorname{gr}\Gamma$ for all $v \in bm\mathsf{S}$, and also continuous when $v \in bc\mathsf{S}$.

Most standard cases from the literature are regular, including all dynamic programs with a finite state space.[7] Further discussion of regularity is provided in Section 3.

Let $\beta_t = \beta(Z_t) \geqslant 0$ for some $\beta \in bm\mathsf{Z}$ and Markov process $\{Z_t\}$ on Z with transition kernel Q.[8] Let $\mathbb{E}_z$ represent expectation given $Z_0 = z$. We call (β, Q) *eventually discounting* if $r_n^\beta < 1$ for some $n \in \mathbb{N}$, where

$$r_n^\beta := \sup_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{t=0}^{n-1} \beta_t.$$

Example 2.1. If there exists a constant $b \geqslant 0$ such that $\beta_t \equiv b$ for all $t \geqslant 0$, then $r_n^\beta = b^n$. Eventual discounting holds if and only if $b < 1$.

Example 2.2. If $\{Z_t\}$ is IID, then $r_n^\beta = \prod_{t=0}^{n-1} \mathbb{E}\beta_t = b^n$ where $b := \mathbb{E}\beta_t$. Hence eventual discounting holds if and only if $\mathbb{E}\beta_t < 1$. In particular, higher moments have no influence on eventual discounting unless there is persistence.

Section 4 provides an extended discussion of eventual discounting for more sophisticated state processes.

Assumption 2.1 (Eventual Contractivity)**.** There is a nonnegative function β in $bc\mathsf{Z}$ and a Feller transition kernel Q on Z such that (β, Q) is eventually discounting and

$$|H(x,z,x',v) - H(x,z,x',w)| \leqslant \beta(z) \int |v(x',z') - w(x',z')| Q(z, dz') \tag{8}$$

[7] The continuity and compactness conditions are automatically satisfied when X and Z are finite and endowed with the discrete topology.

[8] That is, $Q(z,B) = \mathbb{P}\{Z_{t+1} \in B \mid Z_t = z\}$ for all $z \in \mathsf{Z}$ and B in the Borel subsets of Z.

for all $v, w \in bm\mathsf{S}$ and $(x, z, x') \in \operatorname{gr} \Gamma$.[9]

The Feller property means that either Z is discrete or the law of motion is continuous.[10]

2.4. **Optimality Results.** In the statement of the next theorem, a map M from a metric space into itself is called *eventually contracting* if there exists an n in $\mathbb{N}$ such that the n-th iterate M^n is a contraction mapping.[11]

Theorem 2.1. *Let $\mathcal{D}$ be a dynamic program. If $\mathcal{D}$ is regular and Assumption 2.1 holds, then the following statements are true:*

(a) *T_σ is eventually contracting on $bm\mathsf{S}$ and T is eventually contracting on $bc\mathsf{S}$.*

(b) *For each feasible policy σ, the lifetime value v_σ is a well defined element of $bm\mathsf{S}$.*

(c) *The value function v^* is finite, continuous, and the only fixed point of T in $bc\mathsf{S}$.*

(d) *At least one optimal policy exists.*

(e) *Bellman's principle of optimality holds.*

In addition, value function and Howard policy iteration converge:

(f) $\lim_{k\to\infty} T^k v = v^*$ *for all* $v \in bc\mathsf{S}$ *and*

(g) $\lim_{k\to\infty} v_{\sigma_k} = v^*$ *when* $\sigma_k \in \Sigma$ *and* $T_{\sigma_k} v_{\sigma_{k-1}} = T v_{\sigma_{k-1}}$ *for all* $k \in \mathbb{N}$.

This theorem extends the core results of dynamic programming theory to the case of state-dependent discounting. In particular, the value function satisfies the Bellman equation, an optimal policy exists, and Bellman's principle of optimality is valid. Value iteration and policy iteration both lead to the value function, so that we have both existence of an optimal policy and means to compute it. The proof of Theorem 2.1 can be found in the appendix.

Relative to the results that can be obtained under standard contraction conditions (see, e.g., Bertsekas (2013)), the only significant weakening of the main findings is that T

[9]Here we implicitly assume that the discount factor is known to the agent at the beginning of each period. Our results hold for alternative timing with slight modifications to (8). See Section 6.1.

[10]More precisely, we assume that, for any $h \in bc\mathsf{S}$, the function $(x, z) \mapsto \int h(x, z')Q(z, dz')$ is continuous. This holds automatically when Z is countable (under the discrete topology). It also holds if Q is generated by a continuous law of motion, in the sense that $Z_{t+1} = F(Z_t, W_{t+1})$ for some continuous function F and IID sequence $\{W_t\}$. These two cases cover all the applications we consider. Further discussion can be found in Lemma 12.14 of Stokey et al. (1989).

[11]More precisely, a self-map M on metric space (Y, ρ) is called eventually contracting if there exists an n in $\mathbb{N}$ and a $\lambda < 1$ such that $\rho(M^n y, M^n y') \leqslant \lambda \rho(y, y')$ for all y, y' in Y.

and T_σ are eventually contracting, rather than always contracting in one step. Such an outcome cannot be avoided when values of the discount factor greater than one are admitted.

The eventual discounting condition is, in many cases, not just sufficient but also necessary for the dynamic program to be well defined and the optimality results to hold. Appendix A.6 provides additional discussion.

2.5. **Blackwell's Condition.** Blackwell's sufficient condition for a contraction has a natural analogue in the case of state-dependent discounting. As shown in Proposition A.4, if the Bellman operator satisfies

$$[T(v+c)](x,z) \leqslant (Tv)(x,z) + \beta(z) \int c(z') Q(z, dz') \qquad ((x,z) \in \mathsf{S})$$

for all $c \in bm\mathsf{Z}_+$ where (β, Q) is eventually discounting, then T is eventually contracting on $bc\mathsf{S}$. As a consequence, T has a unique fixed point in $bc\mathsf{S}$ that is globally attracting under iteration of T. This extends Blackwell's original result,[12] with the caveat that T might not itself be a contraction. Again, this cannot be avoided when β is allowed to take values greater than one.[13]

2.6. **Monotonicity, Concavity and Differentiability.** Next we show that standard results on monotonicity, concavity, and differentiability of the value function (cf, e.g., Stokey et al. (1989)) are preserved under state-dependent discounting without additional assumptions on the discount factor process. We assume that X is a convex subset of $\mathbb{R}$ in the discussion below and denote $ibc\mathsf{S}$ the set of functions in $bc\mathsf{S}$ that are increasing and concave in x.

Assumption 2.2. For all $v \in ibc\mathsf{S}$ and $z \in \mathsf{Z}$, (i) $x \mapsto H(x, z, x', v)$ is increasing for all $x' \in \Gamma(x,z)$, (ii) $(x, x') \mapsto H(x, z, x', v)$ is strictly concave, (iii) $\Gamma(x,z) \subset \Gamma(y,z)$ for all $x \leqslant y$, and (iv) the set $\{(x,x') : x' \in \Gamma(x,z)\}$ is convex.

Assumption 2.3. The map $x \mapsto H(x, z, x', v)$ is continuously differentiable on $\operatorname{int} \mathsf{X}$ for all $z \in \mathsf{Z}$, $x' \in \operatorname{int} \Gamma(x,z)$, and $v \in ibc\mathsf{S}$.

[12]The original result states that if an operator T is monotone and there exists a $b \in (0,1)$ such that $T(v+c) \leqslant Tv + bc$ for all $c \geqslant 0$, then T is a contraction (see, e.g., Stokey et al., 1989, Theorem 3.3).

[13]In fact, when T is an eventual contraction on a Banach space, one can construct a complete metric on the same space under which T is a contraction. See, for example, Krasnosel'skii et al. (1972). Our terminology on contractions in this section refers specifically to the supremum norm.

The following theorem shows that the value function v^* is increasing, strictly concave, and continuously differentiable in x under standard assumptions.[14]

Theorem 2.2. *If $\mathcal{D}$ is regular and Assumptions 2.1–2.2 hold, then $x \mapsto v^*(x,z)$ is increasing and strictly concave and $x \mapsto \sigma^*(x,z)$ is single-valued and continuous for all $z \in \mathsf{Z}$. If, in addition, Assumption 2.3 holds, then $x \mapsto v^*(x,z_0)$ is continuously differentiable at x_0 whenever $x_0 \in \operatorname{int} \mathsf{X}$ with $\sigma^*(x_0,z_0) \in \operatorname{int} \Gamma(x_0,z_0)$ for some z_0, and*

$$v_x^*(x_0,z_0) = H_x(x_0,z_0,\sigma^*(x_0,z_0),v^*).$$

Additional comments on these assumptions and results can be found in the applications.

3. Additively Separable Problems

In this section we study state-dependent discounting in settings where preferences are additively separable and rewards are bounded. (Extensions to unbounded rewards and recursive preferences are deferred to Sections 5 and 6.)

3.1. An Additively Separable Problem.

Consider the dynamic program in Section 9.2 of Stokey et al. (1989) with the addition of state-dependent discounting. The objective is to maximize

$$\mathbb{E} \sum_{t=0}^{\infty} \prod_{i=0}^{t-1} \beta_i \, F(X_t, Z_t, X_{t+1}) \quad \text{s.t. } X_{t+1} \in \Gamma(X_t, Z_t) \text{ for all } t \geqslant 0. \tag{9}$$

As in Stokey et al. (1989), F is assumed to be bounded and continuous on $\operatorname{gr}\Gamma$, while Γ is a continuous, nonempty, and compact-valued correspondence. We set $\beta_t = \beta(Z_t)$ where β is continuous, bounded and nonnegative, while $\{Z_t\}$ is Markov with Feller kernel Q.

We connect this dynamic program to our framework by setting $\mathcal{D} = (\Gamma, H)$ with

$$H(x,z,x',v) := F(x,z,x') + \beta(z) \int v(x',z') Q(z,dz') \tag{10}$$

for all $v \in bm\mathsf{S}$. The monotonicity condition (2) is clearly satisfied. The function $(x,z,x') \mapsto H(x,z,x',v)$ is bounded and Borel measurable on $\operatorname{gr}\Gamma$ because v and F have these properties, and continuous when v is continuous by the Feller property (see footnote 10). Hence $\mathcal{D}$ is regular.

[14] If $\mathcal{D}$ is additively separable, sufficiency of the Euler equations and transversality conditions can also be established, analogous to Section 9.5 of Stokey et al. (1989).

If (β, Q) is eventually discounting then Assumption 2.1 holds, since (10) yields

$$|H(x,z,x',v) - H(x,z,x',w)| \leqslant \beta(z) \left| \int [v(x',z') - w(x',z')] Q(z,dz') \right|,$$

and an application of the triangle inequality gives (8).

To connect this application with the definition of optimality given in Section 2.2, fix $\sigma \in \Sigma$ and $v \in bm\mathsf{S}$. The policy operator T_σ from (4) can be expressed as

$$(T_\sigma v)(x_0, z_0) = F(x_0, z_0, \sigma(x_0, z_0)) + \beta(z_0)\, \mathbb{E}_0\, v(X_1, Z_1) \tag{11}$$

where $\{X_t\}$ is generated by $X_{t+1} = \sigma(X_t, Z_t)$, the initial condition is $(X_0, Z_0) = (x_0, z_0)$, and $\mathbb{E}_t$ conditions on $\{Z_i\}_{i \leqslant t}$. If we take T_σ, iterate forward n times and apply the law of iterated expectations, we obtain

$$(T_\sigma^n v)(x_0, z_0) = \mathbb{E}_0 \sum_{t=0}^{n-1} \prod_{i=0}^{t-1} \beta_i F(X_t, Z_t, X_{t+1}) + \mathbb{E}_0 \prod_{i=0}^{n} \beta_i v(X_n, Z_n). \tag{12}$$

Recall from (6) that, to obtain the value v_σ of the policy σ, we take the limit of (12) in n. Eventual discounting implies that the second term vanishes as $n \to \infty$.[15] In the limit we obtain as v_σ the value in (9) under the policy σ. Maximizing over σ in Σ yields the optimal policy.

The Bellman operator corresponding to $\mathcal{D}$ is the map T on $bc\mathsf{S}$ defined by

$$(Tv)(x,z) = \max_{x' \in \Gamma(x,z)} \left\{ F(x,z,x') + \beta(z) \int v(x',z') Q(z,dz') \right\}. \tag{13}$$

Since the conditions of Theorem 2.1 are satisfied, the unique fixed point of T in $bc\mathsf{S}$ is $v^* := \sup_{\sigma \in \Sigma} v_\sigma$, the value function of $\mathcal{D}$. Bellman's principle of optimality applies and an optimal policy can be computed by either value function iteration or Howard's policy iteration algorithm. Monotonicity, concavity and differentiability of v^* can be obtained by imposing the same conditions that Stokey et al. (1989) impose on F and Γ and then applying Theorem 2.2.

3.2. **Application to a Savings Problem.** The dynamic program associated with the household problem in Hubmer et al. (2020) can be placed with the framework provided in the previous section. The continuation aggregator takes the form

$$H(x,z,x',v) = u(R(x,z)x + y(x,z) - x') + \beta(z) \int v(x',z') Q(z,dz') \tag{14}$$

[15]This term is dominated by $r^\beta_{n+1} \|v\|$. Hence it suffices to prove that $r^\beta_n \to 0$ as $n \to \infty$. Eventual discounting implies that $r^\beta_n < 1$ for some n, and, as shown in Proposition 4.1 below, this in turn gives $\lim_{n\to\infty} (r^\beta_n)^{1/n} < 1$. But then $r^\beta_n \to 0$, as was to be shown.

where $x \in \mathsf{X} := \mathbb{R}_+$ is current assets, z is a vector of exogenous shocks taking values in $\mathbb{R}^k$, $R(x, z)$ is the gross rate of return on asset holdings (which depends on both exogenous shocks and current asset holdings) and $y(x, z)$ is labor income net of income tax and capital gains tax, as well as a lump sum transfer. The utility function is

$$u(c) := \frac{c^{1-\gamma}}{1-\gamma} \text{ where } \gamma > 1. \tag{15}$$

Next period assets x' are constrained to lie in

$$\Gamma(x, z) := \{x' \in \mathbb{R} : \bar{x} \leqslant x' \leqslant R(x, z)x + y(x, z)\}. \tag{16}$$

This problem is not regular because H is not bounded, since u is unbounded below. However, in solving this dynamic program, Hubmer et al. (2020) reduce both the asset space X and the exogenous shock space Z to a finite grid. The aggregator is then bounded and the continuity parts of the regularity condition are automatically satisfied (under the discrete topology). Hence, to show that all of the conclusions of Theorem 2.1 apply, we need only verify that eventual discounting holds. This issue is discussed for the parameterization in Hubmer et al. (2020) in Section 4 below.

4. The Discount Condition

In this section we discuss tests for the eventual discounting condition and develop intuition regarding its value.

4.1. **Connection to Spectral Radii.** Given β and Q as in Assumption 2.1, let $L_\beta \colon bm\mathsf{Z} \to bm\mathsf{Z}$ be the *discount operator* defined by

$$(L_\beta h)(z) = \beta(z) \int h(z') Q(z, dz') \qquad (h \in bm\mathsf{Z},\ z \in \mathsf{Z}). \tag{17}$$

The next proposition shows that we can test Assumption 2.1 by computing the spectral radius $r(L_\beta)$ of the operator L_β.[16] In stating it, we set $\beta_t := \beta(Z_t)$ where $\{Z_t\}$ is a Z-valued Markov process generated by Q.

Proposition 4.1. *The spectral radius of L_β satisfies $r(L_\beta) = \lim_{n\to\infty}(r_n^\beta)^{1/n}$. Moreover, (β, Q) is eventually discounting if and only if $r(L_\beta) < 1$.*

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[16] As usual, the spectral radius of a bounded linear operator L from a Banach space B to itself is given by $r(L) := \lim_{n\to\infty} \|L^n\|^{1/n}$, where $\|\cdot\|$ is the operator norm. This limit always exists and is equal to $\inf_{n\in\mathbb{N}} \|L^n\|^{1/n}$. If B is finite dimensional, it equals the maximal modulus of the eigenvalues of L. See, for example, Bühler and Salamon (2018), Theorem 1.5.5.

The expression for $r(L_\beta)$ in Proposition 4.1 is obtained through a local spectral radius condition for positive linear operators. It provides both a simple representation of the spectral radius of L_β and a link to eventual discounting. For example, it is immediate from $r(L_\beta) = \lim_{n\to\infty}(r_n^\beta)^{1/n}$ that $r_n^\beta \to 0$ when $r(L_\beta) < 1$. This, in turn, implies that (β, Q) is eventually discounting. The converse implication is more subtle and involves the Markov property. Details are in the appendix.

4.2. **Finite Exogenous State.** Testing eventual discounting is simple when Z is finite. In this case, Q can be represented as a Markov matrix of values Q_{ij}, giving the one-step probability of transitioning from z_i to z_j, and L_β can be represented as the matrix

$$L_\beta := (\beta_i Q_{ij})_{1 \leqslant i,j \leqslant N} \,. \tag{18}$$

Here $\beta_i := \beta(z_i)$ and N is the number of elements in Z. The spectral radius $r(L_\beta)$ is equal to the dominant eigenvalue of L_β, which is real and nonnegative by the Perron–Frobenius Theorem. In view of Proposition 4.1, eventual discounting holds if and only if this eigenvalue is strictly less than unity.

Example 4.1. Christiano et al. (2011) consider the case $\beta_t \in \{\beta^\ell, \beta^h\}$ with $\beta^\ell < 1 < \beta^h$. The process $\{\beta_t\}$ stays at β^h with probability p and shifts permanently to β^ℓ with probability $1-p$. Thus, by (18),

$$L_\beta = \begin{pmatrix} \beta^\ell & 0 \\ (1-p)\beta^h & p\beta^h \end{pmatrix}.$$

The eigenvalues are β^ℓ and $p\beta^h$, so $r(L_\beta)$ is the maximum of these values. Since $\beta^\ell < 1$, eventual discounting holds if and only if $p\beta^h < 1$. The condition is violated if the state β^h is too large or too persistent. Christiano et al. (2011) set $\beta^h = 1.02$ and consider $p \leqslant 0.82$, so eventual discounting is satisfied. Since their household problem can be treated in the same way as in Section 3.2, all of our results in Section 2.4 apply.

4.3. **Stationary Spectral Radius.** The expression obtained for $r(L_\beta)$ in Proposition 4.1 is a geometric mean, and hence is determined by the asymptotic behavior of the discount process. When $\{Z_t\}$ is irreducible, it seems likely that these asymptotics will be independent of the initial condition z. This suggests that the conditional expectation and supremum in the definition of r_n^β can be replaced by the unconditional expectation $\mathbb{E}$ for the stationary process. The next proposition confirms this intuition.

Proposition 4.2. *If Z is finite and the exogenous state process $\{Z_t\}$ is irreducible, then $r(L_\beta)$ satisfies the stationary representation*

$$r(L_\beta) = s^\beta \quad \textit{where} \quad s^\beta := \lim_{n\to\infty} (s_n^\beta)^{1/n} \quad \textit{with} \quad s_n^\beta := \mathbb{E} \prod_{t=0}^{n-1} \beta_t. \tag{19}$$

Our analysis below shows that this stationary representation is also highly accurate even when Z is infinite, provided that $\{Z_t\}$ is irreducible and sufficiently mean reverting for dependence on initial conditions to die out. This is helpful because the stationary representation of $r(L_\beta)$ sometimes admits analytical solutions that facilitate benchmark calculations and enhance intuition.[17]

4.4. **Autoregressive Specifications.** Some studies adopt discount processes that are autoregressive in levels or logs (e.g., Hubmer et al., 2020; Hills et al., 2019; Nakata, 2016) and then discretize them prior to computation. Such specifications always fit the dynamic programming framework adopted above after discretization.[18] The only remaining issue is whether or not eventual discounting holds. For common reference, all examples use the state process

$$Z_{t+1} = \rho Z_t + (1-\rho)\mu + \sigma_\epsilon \epsilon_{t+1}, \quad \{\epsilon_t\} \overset{\text{IID}}{\sim} N(0,1). \tag{20}$$

4.4.1. *AR(1) in Levels.* We first give examples where β_t is a multiple of Z_t. After following the discretization procedure used by the authors, we calculate the spectral radius of the matrix (18).

Example 4.2. Hubmer et al. (2020) take the AR(1) specification $\beta_t = Z_t$ where $\{Z_t\}$ follows (20) with $\rho = 0.992$, $\mu = 0.944$ and $\sigma_\epsilon = 0.0006$ and discretize the process onto a grid of 15 states via Tauchen's method. This gives $r(L_\beta) = 0.9469$, so eventual discounting holds. This is as expected, since the mean μ is substantially less than one and low volatility suggests that the impact of stochastic variation is minor.

Example 4.3. In Hills et al. (2019), the discount process is $\beta_t = bZ_t$ where $\{Z_t\}$ obeys (20). They consider several parameterizations, the most empirically motivated of which is $\mu = 1$, $b = 0.99875$, $\rho = 0.85$ and $\sigma_\epsilon = 0.0062$. Under this parameterization

[17] While finiteness of the state space can be weakened, as discussed above, irreducibility is essential. To see this, consider the application in Christiano et al. (2011), where the unique stationary distribution puts all mass on the low state and irreducibility fails. With all mass on the low state we have $s_n^\beta = (\beta^\ell)^n$ for all n, and hence $s^\beta = \beta^\ell$, which differs from $r(L_\beta) = \max\{\beta^\ell, p\beta^h\}$.

[18] Recall that β is assumed to be bounded and continuous in Assumption 2.1. Both conditions hold after discretization. (Continuity holds automatically under the discrete topology.)

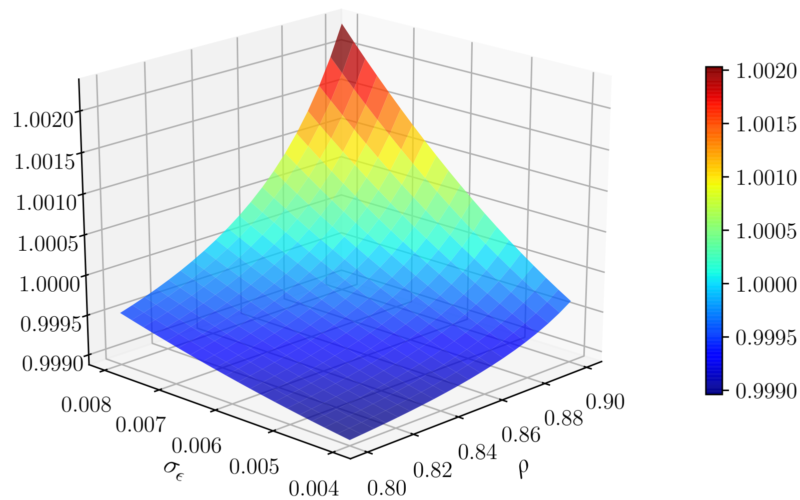

FIGURE 2. $r(L_\beta)$ as a function of ρ and σ_ϵ; $\mu = 0.944$

β_t regularly exceeds one, as observed in the simulated process shown in Figure 1. Nonetheless, after following their discretization procedure and computing the spectral radius of L_β, we find $r(L_\beta) = 0.9996$, so eventual discounting holds.

Example 4.4. In a similar setting to Example 4.3, Nakata (2016) assumes $\beta_t = bZ_t$ where $\{Z_t\}$ follows (20), $\mu = 1$, $b = 0.995$, $\rho = 0.85$, and $\sigma_\epsilon = 0.00395$. The process is discretized onto a grid of 501 points, yielding $r(L_\beta) = 0.9953$.

To illustrate how the stochastic properties of β_t affect the size of $r(L_\beta)$, we take the parameterization in Example 4.3 as a benchmark and vary the persistence term ρ and the volatility σ_ϵ. Other parameters are held constant. Figure 2 plots the resulting values of $r(L_\beta)$. The figure shows that higher volatility and higher persistence both increase $r(L_\beta)$, leading to a failure of eventual discounting when $r(L_\beta) \geqslant 1$. Note also that there is a positive interaction between persistence and volatility, with the effect of each parameter enhanced by the other.

Some further insight can be gained by considering the expected two period discount factor when $\beta_t = Z_t$ and $\{Z_t\}$ is as given in (20). Under the stationary distribution, which governs asymptotic outcomes, this evaluates to

$$\mathbb{E}\beta_t\beta_{t+1} = \mu^2 + \rho\frac{\sigma_\epsilon^2}{1-\rho^2}. \tag{21}$$

The value in (21) depends on the sign of ρ. Positive correlation combined with positive volatility in the state process leads to a value greater than the stationary mean. This

TABLE 1. Comparison of s^β and $r(L_\beta)$ after discretization

Parameters		N=10		N=200	
$\mu = -0.05$	s^β	$r(L_\beta)$	Error	$r(L_\beta)$	Error
$\rho = 0.90,\ \sigma_\epsilon = 0.01$	0.956	0.956	2.5e-05	0.956	1.1e-06
$\rho = 0.90,\ \sigma_\epsilon = 0.02$	0.970	0.970	3.9e-04	0.970	1.8e-05
$\rho = 0.92,\ \sigma_\epsilon = 0.01$	0.959	0.959	7.6e-05	0.959	3.5e-06
$\rho = 0.92,\ \sigma_\epsilon = 0.02$	0.981	0.980	1.2e-03	0.981	5.8e-05
$\rho = 0.94,\ \sigma_\epsilon = 0.01$	0.965	0.964	3.2e-04	0.965	1.5e-05
$\rho = 0.94,\ \sigma_\epsilon = 0.02$	1.006	1.001	4.7e-03	1.005	2.5e-04

is because, under positive correlation, positive deviations from the mean tend to occur consecutively and reinforce each other.

4.4.2. *AR(1) in Logs.* Next we set $\beta_t := \exp(Z_t)$ where $\{Z_t\}$ obeys the AR(1) specification (20). This specification is arguably more natural than the direct AR(1) approach discussed above due to positivity. While the state space is not finite, irreducibility of $\{Z_t\}$ leads us to conjecture that an approximate version of Proposition 4.2 holds, so that the stationary geometric mean $s^\beta = \lim_{n\to\infty}(s_n^\beta)^{1/n}$ for the original process will be close to $r(L_\beta) = \lim_{n\to\infty}(r_n^\beta)^{1/n}$ when the latter is calculated using an appropriately discretized version of the process. As shown in Appendix A.5, for the original process we have

$$s^\beta = \lim_{n\to\infty}(s_n^\beta)^{1/n} = \lim_{n\to\infty}\left(\mathbb{E}\prod_{t=0}^{n-1}\beta_t\right)^{1/n} = \exp\left\{\mu + \frac{\sigma_\epsilon^2}{2(1-\rho)^2}\right\}. \tag{22}$$

Numerical experiments show that the expression on the right hand side of (22) provides a good approximation of $r(L_\beta)$ even when the discretization is relatively coarse, and an almost perfect approximation when the discretization is fine. Table 1 illustrates by comparing s^β given by (22) and $r(L_\beta)$ under two different levels of discretization, for a range of parameter values.[19]

Given this tight relationship between s^β and $r(L_\beta)$, we can use (22) to examine how the parameters of the state process affect eventual discounting. Consistent with our previous findings, the expression in (22) indicates that $r(L_\beta)$ is increasing in all of the three parameters (although the effect is now exponential). Higher persistence and

[19] N is the number of grid points. We use the Rouwenhorst's method for discretization, which has strong asymptotic properties in terms of approximating the distributions of Gaussian AR(1) processes (Kopecky and Suen, 2010). We fix μ because it has no effect on the errors.

higher volatility reinforce each other. The impact of ρ is nonlinear and large in the neighborhood of unity.

5. Unbounded Rewards

In this section we show that the optimality results presented above extend to a range of unbounded reward settings after suitable modifications. We consider the additively separable aggregator

$$H(x, z, x', v) = u(x, z, x') + \beta(z) \int v(x', z') Q(z, dz'). \tag{23}$$

The continuation value function v is in $\mathcal{V}$, which is the set of all *candidate value functions* and varies across applications. As before, $\beta \in bc\mathsf{Z}$ and Q is a Feller transition kernel. The feasible correspondence Γ is assumed to be continuous, nonempty, and compact valued. The reward function u is continuous but not necessarily bounded. The Euclidean norm is represented by $|\cdot|$.

5.1. **Homogeneous Functions.** We begin by extending the core results of Alvarez and Stokey (1998) to the case of state-dependent discounting. We consider reward functions that are homogeneous of degree $\theta \in (0, 1]$ and feasible correspondences that are homogeneous of degree one.[20]

Assumption 5.1. X is a convex cone in $\mathbb{R}^k_+$ and $\lambda x' \in \Gamma(\lambda x, z)$ when $(x, z, x') \in \operatorname{gr} \Gamma$ and $\lambda \geqslant 0$. For each $z \in \mathsf{Z}$, $u(\cdot, z, \cdot)$ is homogeneous of degree θ, and there exists a $B > 0$ such that

$$|u(x, z, x')| \leqslant B(|x| + |x'|)^\theta \text{ for all } (x, z, x') \in \operatorname{gr} \Gamma.$$

Assumption 5.1 follows Alvarez and Stokey (1998). The next assumption generalizes their growth restriction to problems with state-dependent discounting.

Assumption 5.2. There exists an $\alpha \geqslant 0$ in $bm\mathsf{Z}$ such that $|x'| \leqslant \alpha(z)|x|$ when $(x, z, x') \in \operatorname{gr} \Gamma$. In addition, for $\{Z_t\}$ generated by Q,

$$\sup_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{t=0}^{n-1} \beta(Z_t) \alpha^\theta(Z_t) < 1 \text{ for some } n \in \mathbb{N}. \tag{24}$$

[20]Recall that a real-valued f defined on a convex cone C of $\mathbb{R}^k$ is homogeneous of degree θ if $f(\lambda x) = \lambda^\theta f(x)$ for all $\lambda \geqslant 0$ and $x \in C$.

The function α is a state-dependent upper bound on the growth rate of the state variable. Comparing to the eventual discounting condition in Section 2.3, the extra term $\alpha^\theta(Z_t)$ in (24) reflects the need to take into account the growth restriction when the reward function is homogeneous and unbounded above. If both β and α are constant, then (24) reduces to the condition $\alpha^\theta \beta < 1$ used in Alvarez and Stokey (1998).

In household problems where the state is asset holdings, the gross asset return bounds the growth rate of the state. The condition in (24) implies that the shocks to the discount factor and asset return have a similar effect on eventual discounting, but their relative importance depends on the degree of homogeneity of the reward function.

Let $(h_\theta \mathsf{S}, \|\cdot\|_h)$ be the space of continuous functions on S that are homogeneous of degree θ in x and bounded in the norm defined by

$$\|f\|_h := \sup\{|f(x,z)| : z \in \mathsf{Z},\, x \in \mathsf{X},\, |x| = 1\}. \tag{25}$$

Then $h_\theta \mathsf{S}$ is a Banach space (Stokey et al., 1989). To make the problem well defined, we let $v_0 \equiv \mathbf{0}$ so the σ-value function is given by $v_\sigma := \lim_n (T_\sigma \mathbf{0})$.

Proposition 5.1. *Let $\mathcal{V} = h_\theta \mathsf{S}$. Under Assumptions 5.1–5.2, the lifetime value v_σ is well defined and finite on S for any feasible policy σ, the value function v^* is a unique fixed point of T on $\mathcal{V}$, $T^n v \to v^*$ for all $v \in \mathcal{V}$, there exists an optimal policy that is homogeneous of degree one, and the principle of optimality holds.*

Example 5.1. Consider the household saving problem in Toda (2019) where the exogenous state $\{Z_t\}$ is Markovian on Z with stochastic kernel Q. The asset return R and discount function β are bounded continuous functions of Z_t. The utility function is $u(c) = c^{1-\gamma}/(1-\gamma)$ with $\gamma \in (0,1)$. The budget constraint is $X_{t+1} = R(Z_t)(X_t - C_t) \geqslant 0$ where X_t is the beginning-of-period wealth and C_t is consumption. The Bellman equation is

$$v(x,z) = \max_{c, x' \geqslant 0} \left\{ u(c) + \beta(z) \int v(x', z')\, Q(z, dz') \right\} \quad \text{s.t.} \quad x' = R(z)(x - c).$$

If we use the constraint to eliminate c in the Bellman equation and let $\Gamma(x,z) = [0, R(z)x]$, then Assumption 5.1 is satisfied with $\theta = 1 - \gamma$ and $B = 1/(1-\gamma)$. By Proposition 4.1, Assumption 5.2 holds if $r(L_\alpha) < 1$ with L_α defined by

$$(L_\alpha h)(z) := \beta(z) R^{1-\gamma}(z) \int h(z') Q(z, dz'),$$

where we let the upper bound function $\alpha = R$. This is a direct extension of the results in Toda (2019) to the case of infinite Z. In particular, the condition $r(L_\alpha) < 1$ reduces to the condition in Proposition 1 of Toda (2019) whenever Z is finite.

5.2. **Local Contractions.** Next we adopt a local contraction approach to dynamic programs with state dependent discounting and unbounded rewards, extending methods first developed in Rincón-Zapatero and Rodríguez-Palmero (2003). As in the previous section, the aggregator has the form of (23).

Let $c\mathsf{S}$ be all continuous functions on S. Let Z be compact and write $\mathsf{X} = \bigcup_j \operatorname{int} K_j$ where $\{K_j\}$ is a sequence of strictly increasing and compact subsets of X. Let

$$\|f\|_j := \sup_{x \in K_j, z \in \mathsf{Z}} |f(x,z)| \qquad (f \in c\mathsf{S}).$$

Let $c > 1$ and $\{m_j\}$ be an unbounded sequence of increasing positive real numbers. Let $c_m\mathsf{S}$ be all $f \in c\mathsf{S}$ such that

$$\|f\|_m := \sum_{j=1}^{\infty} \frac{\|f\|_j}{m_j c^j} < \infty.$$

The pair $(c_m\mathsf{S}, \|\cdot\|_m)$ forms a Banach space (Matkowski and Nowak, 2011).

Assumption 5.3. $\Gamma(x,z) \subset K_j$ for all $x \in K_j$, all $z \in \mathsf{Z}$, and all $j \in \mathbb{N}$, and (β, Q) is eventually discounting in the sense of Section 2.3.

Proposition 5.2. *Under Assumption 5.3, the lifetime value v_σ is well defined and finite on S for any $\sigma \in \Sigma$, there exists a sequence $m_j \uparrow \infty$ such that the value function v^* is the unique fixed point of T on $c_m\mathsf{S}$, $T^n v \to v^*$ for all $v \in c_m\mathsf{S}$, there exists an optimal policy, and the principle of optimality holds.*

Example 5.2. Consider a stochastic optimal growth model with state dependent discounting, total production $zf(x)$ and continuous utility u. The feasible correspondence is $\Gamma(x,z) = [0, zf(x)]$. Let $\mathsf{X} = \mathbb{R}_+$ and let $\mathsf{Z} \subset \mathbb{R}_+$ be compact. Suppose $f' > 0$, $f'' < 0$ and $\lim_{x\to\infty} f'(x) = 0$. Let $\{K_j\}$ be an increasing sequence of compact sets covering X such that $\Gamma(x,z) \subset K_j$ for all $x \in K_j$.[21] Assumption 5.3 holds and Proposition 5.2 can be applied if (β, Q) is eventual discounting.

6. Further Extensions

We study two further extensions. Section 6.1 studies an alternative discount specification to the framework in Section 2. Section 6.2 extends our main results to Epstein-Zin preferences with unbounded rewards.

[21] For example, set $K_j := [0, M + j]$ for all $j \in \mathbb{N}$, where M is some large constant.

6.1. **Alternative Discount Specifications.** Discounting methods that differ from the preceding framework can also be analyzed. To illustrate, we consider the shocks to long-run discount factors found in Primiceri et al. (2006), Justiniano et al. (2010), Leeper et al. (2010), and Christiano et al. (2014). Their maximization problems are analogous to the additively separable problem in Section 3.1, with the difference that $\prod_{t=0}^{n-1} \beta_t$ is replaced by $b^n Z_n$ for some constant b. While the discount factor $b^n Z_n$ can be expressed as $\prod_{t=0}^{n-1} \beta_t$ after setting $\beta_t := bZ_{t+1}/Z_t$ and $Z_0 = 1$, notice that β_t is not observable until $t+1$. Hence inequality (8) cannot be used, since it assumes that β_t is visible at t.

To handle such cases, one option is to replace inequality (8) with

$$|H(x, z, x', v) - H(x, z, x', w)| \leqslant \int \beta(z')|v(x', z') - w(x', z')|Q(z, dz'). \tag{26}$$

Inequality (26) integrates over $\beta(z')$, supposing that its realization is not observed at the time that x' is chosen. We prove in the appendix that Theorem 2.1 extends to this case: the theorem is valid under eventual discounting when (26) replaces (8).

The set up of Primiceri et al. (2006) and other authors mentioned above satisfies (26) after redefining the aggregator and the exogenous state variable.[22] The only question, then, is whether or not eventual discounting holds. The following proposition shows that, in many cases, the answer depends only on the value of b in $\beta_t := bZ_{t+1}/Z_t$. Stochastic components are irrelevant.

Proposition 6.1. *If $\beta_t := bZ_{t+1}/Z_t$ for all t and $\{Z_t\}$ is positive and bounded, then eventual discounting holds if and only if $b < 1$.*

The intuition behind Proposition 6.1 is that the spectral radius $r(L_\beta)$ equals the asymptotic growth rate of the discount factor process. If $\prod_{t=0}^{n-1} \beta_t = b^n Z_n$ and Z_t is positive and bounded, the asymptotic growth rate is equal to b.

6.2. **Epstein-Zin Preferences.** Next we extend the preceding results on dynamic programming under state-dependent discounting to settings where lifetime utility is governed by Epstein–Zin preferences. Lifetime utility of an agent satisfies

$$U(C_t, C_{t+1}, \dots) = \left\{ C_t^{1-1/\psi} + \beta_t \left[\mathbb{E}_t U^{1-\gamma}(C_{t+1}, C_{t+2}, \dots) \right]^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}, \tag{27}$$

[22]To be specific, let the exogenous state variable be $\tilde{Z}_{t+1} = (Z_{t+1}, Z_t)$. The aggregator then becomes $H(x, z, x', v) = \tilde{F}(x, z, x') + \int \beta(z')v(x', z')\tilde{Q}(z, dz')$, where $\tilde{F}(X_t, \tilde{Z}_t, X_{t+1}) = F(X_t, Z_t, X_{t+1})$, $\beta(\tilde{Z}_{t+1}) = bZ_{t+1}/Z_t$, and $\tilde{Q}$ is the transition kernel on $\tilde{\mathsf{Z}} := \mathsf{Z}^2$ induced by Q.

where γ is the relative risk aversion and ψ is the elasticity of intertemporal substitution. The agent maximizes lifetime utility by choosing consumption $\{C_t\}$ subject to $X_{t+1} = R_t(X_t - C_t) \geqslant 0$. Here X_t is asset holding of the agent at the beginning of time t and R_t is returns. We focus on the empirically relevant case of $\gamma > 1$ and $\psi > 1$, as in, say, Bansal and Yaron (2004), Albuquerque et al. (2016), or Schorfheide et al. (2018). This is the most challenging setting because the usual contraction argument fails and the utility function is unbounded above.

6.2.1. *Discounting Continuation Values.* Let $\mathsf{X} = \mathbb{R}_+$, assume that β_t and R_t are functions of the exogenous state, and define the aggregator H by

$$H(x, z, c, v) = \left\{ c^{1-1/\psi} + \beta(z) \left[\int v\left(R(z)(x-c), z'\right)^{1-\gamma} Q(z, dz') \right]^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}, \quad (28)$$

where x, z, and c are asset holding, exogenous state, and consumption, respectively, satisfying $c \in \Gamma(x, z) = [0, x]$.

Assumption 6.1. The functions β and R are nonnegative elements of $bm\mathsf{Z}$. In addition, for $\{Z_t\}$ generated by Q, we have

$$\sup_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{t=0}^{n-1} \beta(Z_t)^{1/(1-1/\psi)} R(Z_t) < 1 \quad \text{for some } n \in \mathbb{N}. \quad (29)$$

Assumption 6.1 is an eventual discounting condition for the Epstein–Zin case. It is modified to take into account both the underlying growth rate, as in Assumption 5.2, and also the role of elasticity of intertemporal substitution. (Intuition and numerical applications are provided below.)

Let $\mathcal{V}$ be all $f \in m\mathsf{S}$ such that $\|f\|_I := \sup_{x \in \mathsf{X}, z \in \mathsf{Z}} |f(x, z)/(1 + x)|$ is finite. We show in Appendix A.4.2 that there exists an upper bound function $\hat{v} \in \mathcal{V}$ such that T_σ is a self map on the order interval $[0, \hat{v}] \subset \mathcal{V}$ with the pointwise partial order. Then we show that $v_\sigma := \lim_n (T_\sigma^n \mathbf{0})$ is well defined on the order interval and is a fixed point of T_σ. In addition, if σ satisfies an interiority condition, the fixed point is unique. See Proposition A.6.

Let $\hat{\mathcal{V}}$ be the space of functions in $\mathcal{V}$ that are homogeneous of degree one in x. Our main result for this section is as follows.

Proposition 6.2. *If Assumption 6.1 holds, then $\bar{v} := \lim_{n \to \infty} T^n \mathbf{0}$ is a well defined element of $\hat{\mathcal{V}}$ and equal to the value function. There exists an optimal policy $\sigma^* \in \Sigma$ that is homogeneous of degree one in x and the principle of optimality holds.*

Notice that Proposition 6.2 contains no analogue of the eventual contraction condition in Assumption 2.1. This is because, as mentioned above, T and T_σ are not contraction mappings under conventional metrics. Instead, the proof uses monotonicity and a form of concavity inherent in Epstein–Zin preferences, combined with fixed point results due to Marinacci and Montrucchio (2010).

6.2.2. *Alternative Preference Shocks.* While (27) parallels the definitions in, say, Epstein and Zin (1989), Nakata and Tanaka (2020) and de Groot et al. (2020), other studies introduce preference shocks to current consumption (Albuquerque et al., 2016; Schorfheide et al., 2018). In this setting, lifetime utility satisfies

$$U(C_t, C_{t+1}, \ldots) = \left\{ \lambda_t C_t^{1-1/\psi} + b \left[\mathbb{E}_t U^{1-\gamma}(C_{t+1}, C_{t+2}, \ldots) \right]^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}, \tag{30}$$

where $b < 1$ is a fixed constant and $\{\lambda_t\}$ is a preference shock.[23] As we now show, the preceding analysis can be brought to bear on this case as well.

Using homogeneity and dividing both sides of (30) by $\lambda_t^{1/(1-1/\psi)}$ yields

$$\tilde{U}_t = \left\{ C_t^{1-1/\psi} + b \left[\mathbb{E}_t \tilde{U}_{t+1}^{1-\gamma} \left(\frac{\lambda_{t+1}}{\lambda_t} \right)^{\frac{1-\gamma}{1-1/\psi}} \right]^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}, \tag{31}$$

where $\tilde{U}_t := U(C_t, C_{t+1}, \ldots)/\lambda_t^{1/(1-1/\psi)}$. If λ_{t+1}/λ_t is measurable with respect to the time-t information set, then (31) becomes

$$\tilde{U}_t = \left\{ C_t^{1-1/\psi} + b\delta_t \left[\mathbb{E}_t \tilde{U}_{t+1}^{1-\gamma} \right]^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}, \tag{32}$$

where $\delta_t := \lambda_{t+1}/\lambda_t$. This is the same as the original Koopmans equation in (27) with $\beta_t = b\delta_t$.[24] Optimality results from the previous section can now be applied. In particular, Proposition 6.2 can be directly applied to the agent's problem in Albuquerque et al. (2016).

[23] Some authors also place an additional term $(1-b)$ before λ_t. This is inconsequential to our optimality results since we can simply redefine λ_t to include $(1-b)$.

[24] The equivalence between β_t and $b\lambda_{t+1}/\lambda_t$ is demonstrated in de Groot et al. (2020) using the Euler equation in an expected utility setting.

6.2.3. *Interpretation.* Condition (29) is the key restriction required for Proposition 6.2 and elasticity of intertemporal substitution plays a role. To illustrate the implications of the condition, we consider the study of Albuquerque et al. (2016), who adopt the specification in (30) with $\delta_t := \lambda_{t+1}/\lambda_t$ satisfying $\log \delta_t = \rho \log \delta_{t-1} + \sigma_\epsilon \epsilon_t$. In view of the discussion in Section 6.2.2, we can study optimality by applying the eventual discounting condition (29) to the transformed representation (32). By a result analogous to Proposition 4.1, condition (29) is equivalent to $r(L_R) < 1$ with L_R defined by

$$(L_R h)(z) := \beta(z)^{1/(1-1/\psi)} R(z) \int h(z') Q(z, dz'). \tag{33}$$

One way to obtain insight on the value $r(L_R)$ is to use the stationary approximation $s := \lim_{n\to\infty} s_n^{1/n}$, where $s_n := \mathbb{E} \prod_{t=0}^{n-1} \beta_t^{1/(1-1/\psi)} R_t$. The advantage of the stationary approximation is that, if we specialize to $R(z) \equiv R$, then we obtain the analytical expression

$$s = R \exp\left(\frac{1}{1-1/\psi} \log b + \frac{1}{(1-1/\psi)^2} \frac{\sigma_\epsilon^2}{2(1-\rho)^2} \right). \tag{34}$$

(See Appendix A.5 for details.) Analogous to the findings in Section 4.4.2 (cf. Table 1), this stationary representation closely approximates $r(L_R)$ for a discretized version with moderately fine grid.

The expression in (34) sheds light on the role that elasticity of intertemporal substitution plays in eventual discounting. The impact of ψ in (34) is not monotone because the mean term $\log b$ is typically negative, while the volatility term $\sigma_\epsilon^2/(2(1-\rho)^2)$ is positive. Nonetheless, we can understand the impact of ψ by the relative weight placed on the mean and volatility terms: $1/(1-1/\psi)$ enters (34) directly for the mean and is squared on the volatility term. Hence, as ψ rises and $1/(1-1/\psi)$ falls, the relative importance of b in determining $r(L_R)$ increases. Conversely, as $\psi \downarrow 1$, the volatility term increasingly dominates.

Intuitively, if ψ is large, then the agent is more willing to shift consumption across time, so the volatility in the discount factor plays a lesser role. Conversely, when ψ is small, consumption cannot shift as freely to compensate for fluctuations in the discount factor. Hence volatility in the discount factor has a large impact on lifetime utility.

6.2.4. *Numerical Analysis.* In the applications discussed in Section 4.2, discount dynamics are driven by Gaussian AR(1) processes, where standard discretization methods are available and eventual discounting is easy to test. In some recent studies, however, discounting is driven by a Markov process and additional innovations, as in Albuquerque et al. (2016), or stochastic volatility, as in Basu and Bundick (2017). For

Table 2. Calculate $r(L_R)$ Using Monte Carlo Method

Length of Paths	$n = 100$	$n = 200$	$n = 500$	$n = 1000$
Estimate of $r(L_R)$	1.00355	1.00698	1.01220	1.01321
Standard Error	(0.00004)	(0.00008)	(0.00045)	(0.00054)

such cases, one can either use a more sophisticated discretization procedure (see, e.g., Farmer and Toda (2017)) or use Monte Carlo.

To illustrate the Monte Carlo method, we return to the model in Albuquerque et al. (2016) studied above, where the eventual discounting condition is (29), or equivalently, $r(L_R) < 1$ with L_R defined in (33). An analytical expression was obtained in (34) for the case when R_t is constant, but in Albuquerque et al. (2016) this is not the case. Nonetheless, by the strong law of large numbers, we can approximate each s_n by generating m independent simulated paths of $\{\beta_t, R_t\}$ and calculating

$$\hat{s}_n = \frac{1}{m} \sum_{i=1}^{m} \prod_{t=0}^{n-1} \beta_{i,t}^{1/(1-1/\psi)} R_{i,t}. \tag{35}$$

Using the parameters in Albuquerque et al. (2016), we find that $\hat{s}_n^{1/n}$ increases with n and exceeds one when n is large, as shown in Table 2.[25] This is in line with the analytical expression given by (34), which yields $s = 1.0168$ if we fix $R_t \equiv 1$. Hence eventual discounting fails under their parameterization.[26]

6.2.5. *The Role of Elasticity of Intertemporal Substitution.* In a New Keynesian model with preference similar to (30) studied by Basu and Bundick (2017), de Groot et al.

[25]We treat the baseline model in Albuquerque et al. (2016), where $\gamma = 1.516$ and $\psi = 1.4567$. There are three exogenous states: preference shock x_t, log consumption growth Δc_t, and log price consumption ratio z_{ct}. The discount factor is $\beta_t = b e^{x_t}$ with $x_t = \rho x_{t-1} + \sigma \epsilon_t$, $b = 0.99795$, $\rho = 0.99132$, and $\sigma = 0.00058631$. The logarithm of returns satisfies $r_{t+1} = \kappa_{c0} + \kappa_{c1} z_{ct+1} - z_{ct} + \Delta c_{t+1}$ where $z_{ct} = A_{c0} + A_{c1} x_t$ and $\Delta c_{t+1} = \mu + \sigma_c \epsilon^c_{t+1}$ with $\mu = 0.0015644$ and $\sigma_c = 0.0069004$. The remaining parameters can be solved as detailed in their Internet Appendix, giving $\kappa_{c0} = 0.023108, \kappa_{c1} = 0.99653, A_{c0} = 5.6605$, and $A_{c1} = 82.519$. We run a large number of simulations ($m = 100000$) for each experiment to ensure that $\hat{s}_n$ is close to s_n. The last row lists the standard error for each estimate by calculating the standard deviation of 1000 simulated $\hat{s}_n^{1/n}$ with $\hat{s}_n$ replaced by an approximating normal distribution for computational efficiency.

[26]We have not shown the eventual discounting condition to be necessary in the Epstein–Zin case, so the optimization problem in Albuquerque et al. (2016) might still be well defined. The quantitative exercise in Albuquerque et al. (2016) does not shed light on this issue because they do not solve the agent's optimization problem directly. Instead, they assume that a solution exists and use it to derive asset pricing moments.

(2018) show that the responses to discount factor shocks explode when the elasticity of intertemporal substitution approaches one, and that this issue disappears if β_t is constant. This matches (34). If the volatility term is not zero, then $r(L_R)$ becomes arbitrarily large as ψ approaches one. Hence it appears that the large responses found in de Groot et al. (2018) are the result of an ill-defined household problem that fails the eventual discounting condition. If $\beta_t \equiv b$, then (34) becomes $b^{1/(1-\psi)}R$. Letting ψ approach one will push down $r(L_R)$ instead so the issue disappears.

In de Groot et al. (2018), the asymptote in the responses is attributed to the distributional weights on current and future utility not summing to one. They propose an alternative setting where current utility is weighted by $1 - \beta_t$ and future utility is weighted by β_t with $\beta_t < 1$. We show in the appendix that the eventual discounting condition for this specification is the same as Assumption 6.1. Since β_t is assumed to be strictly less than one in de Groot et al. (2018), we let $\beta_t \leqslant b$ for some $b < 1$ and assume fixed returns. Then (34) implies that $r(L_R) \leqslant b^{1/(1-1/\psi)}R$. The previous discussion shows that, in this case, eventual discounting holds when ψ approaches one. This provides an alternative explanation of why the model does not produce an asymptote in responses to discount factor shocks.

7. Conclusion

We introduce a weak discounting condition and show that, under this condition, standard infinite horizon dynamic programs with state-dependent discount rates are well defined and well behaved. The value function satisfies the Bellman equation, an optimal policy exists, Bellman's principle of optimality is valid, value function iteration converges and so does Howard's policy iteration algorithm. The method can be applied to a broad range of dynamic programming problems, including those with discrete choices, continuous choices and recursive preferences.

We connect eventual discounting to a spectral radius condition and provide guidelines on how to calculate the spectral radius for a range of discount specifications. We show that the condition is more likely to fail when the discount process has higher mean, persistence, or volatility. For models with Epstein–Zin preferences and state-dependent discount factors, the condition also depends on the elasticity of intertemporal substitution.

One natural open question is: how do our results translate into continuous time? It would also be valuable to understand how the results change if discounting depends on endogenous states and actions. Finally, more research is needed on how close to necessary the eventual discounting conditions are for recursive preference models, and especially those involving long run risks, since these models generate realistic asset price processes by driving their parameterizations close to the boundary between stability and instability. These questions are left to future research.

Appendix A. Remaining Proofs

In what follows, we consider the dynamic program described in Section 2.1.

A.1. **Proofs for Section 2.**

A.1.1. *Proof of Theorem 2.1.* For each $\sigma \in \Sigma$, let T_σ be defined on $bm\mathsf{S}$ by (4). Let T be defined on $bc\mathsf{S}$ by (5). We prove part (a) through two lemmas.

Lemma A.1. *If $\sigma \in \Sigma$, then T_σ is eventually contracting on $bm\mathsf{S}$.*

Proof. Fix $\sigma \in \Sigma$ and $v \in bm\mathsf{S}$. The map $T_\sigma v$ is Borel measurable on S by the regularity conditions and measurability of σ. It is bounded by the assumption that H is bounded. Hence T_σ is a self-map on $bm\mathsf{S}$. To see that it is eventually contracting, fix (x, z) in S and observe that, by Assumption 2.1,

$$\begin{aligned} |(T_\sigma v)(x,z) - (T_\sigma w)(x,z)| &= |H(x,z,\sigma(x,z),v) - H(x,z,\sigma(x,z),w)| \\ &\leqslant \beta(z) \int |v(\sigma(x,z),z') - w(\sigma(x,z),z')| Q(z,dz') \end{aligned}$$

for any $v, w \in bm\mathsf{S}$. We can write this expression as

$$|T_\sigma v - T_\sigma w| \leqslant K_\sigma |v - w|, \tag{36}$$

where K_σ is the operator defined by

$$(K_\sigma h)(x,z) := \beta(z) \int h(\sigma(x,z),z') Q(z,dz') \qquad (h \in bm\mathsf{S}, z \in \mathsf{Z}).$$

Since $\beta \in bc\mathsf{Z}$, K_σ is a self-map on $bm\mathsf{S}$. Since K_σ is order preserving, we can iterate on (36) to obtain $|T_\sigma^n v - T_\sigma^n w| \leqslant K_\sigma^n |v - w|$ for all $n \in \mathbb{N}$.

Let $\{Z_t\}$ be a Markov process generated by Q and started at z, let $\beta_t = \beta(Z_t)$, and let $\{X_t\}$ be the controlled Markov process generated by $X_{t+1} = \sigma(X_t, Z_t)$ with $(X_0, Z_0) = (x, z)$. We then have $(K_\sigma h)(x,z) = \mathbb{E}_{x,z}\, \beta_0\, h(X_1, Z_1)$ and, iterating on this equation,

$$(K_\sigma^n h)(x,z) = \mathbb{E}_{x,z}\, \beta_0 \beta_1 \cdots \beta_{n-1}\, h(X_n, Z_n) \leqslant r_n^\beta \|h\|. \tag{37}$$

Since $|T_\sigma^n v - T_\sigma^n w| \leqslant K_\sigma^n |v-w|$, taking the supremum yields $\|T_\sigma^n v - T_\sigma^n w\| \leqslant r_n^\beta \|v - w\|$. It now follows from the eventual discounting property that T_σ^n is a contraction for some $n \in \mathbb{N}$. Hence T_σ is eventually contracting. □

Lemma A.2. *The operator T is eventually contracting on $bc\mathsf{S}$.*

Proof. Fix $v \in bc\mathsf{S}$. The map Tv is continuous on S by regularity and Berge's Maximum Theorem (Aliprantis and Border, 2006, Theorem 17.31). It is bounded by boundedness of H. Hence T is a self-map on $bc\mathsf{S}$. To see that it is eventually contracting, fix (x, z) in S and observe that, by Assumption 2.1,

$$\begin{aligned}|(Tv)(x,z) - (Tw)(x,z)| &\leqslant \max_{x' \in \Gamma(x,z)} |H(x,z,x',v) - H(x,z,x',w)| \\ &\leqslant \max_{x' \in \Gamma(x,z)} \beta(z) \int |v(x',z') - w(x',z')| \, Q(z, dz')\end{aligned}$$

for any $v, w \in bc\mathsf{S}$. We can write this expression as

$$|Tv - Tw| \leqslant K|v - w|, \tag{38}$$

where K is the operator on $bc\mathsf{S}$ defined by

$$(Kh)(x,z) := \max_{x' \in \Gamma(x,z)} \beta(z) \int h(x',z') Q(z,dz') \qquad (h \in bc\mathsf{S}, z \in \mathsf{Z}).$$

It follows from regularity and the Feller property (see footnote 10) that $(x', z) \mapsto \int h(x',z')Q(z,dz')$ is continuous. Since $\beta \in bc\mathsf{Z}$, it follows from the maximum theorem that K is a self-map on $bc\mathsf{S}$. Since K is order preserving, we can iterate on (38) to obtain $|T^n v - T^n w| \leqslant K^n |v - w|$ for all $n \in \mathbb{N}$.

Now set $h := |v - w|$, let $\{Z_t\}$ be a Markov process generated by Q with initial condition z and let $\beta_t = \beta(Z_t)$. We then have $(Kh)(x,z) = \max_{x_1 \in \Gamma(x,z)} \mathbb{E}_z \, \beta_0 \, h(x_1, Z_1)$ and hence

$$\begin{aligned}(K^2 h)(x,z) &= \max_{x_1 \in \Gamma(x,z)} \mathbb{E}_z \, \beta_0 \, (Kh)(x_1, Z_1) \\ &= \max_{x_1 \in \Gamma(x,z)} \mathbb{E}_z \, \beta_0 \max_{x_2 \in \Gamma(x_1, Z_1)} \mathbb{E}_{Z_1} \, \beta_1 \, h(x_2, Z_2) \\ &\leqslant \|h\| \, \mathbb{E}_z \, \beta_0 \beta_1.\end{aligned}$$

More generally, for arbitrary $n \in \mathbb{N}$, we have $(K^n h)(x,z) \leqslant r_n^\beta \|h\|$. Since $|T^n v - T^n w| \leqslant K^n h$, taking the supremum gives $\|T^n v - T^n w\| \leqslant r_n^\beta \|v - w\|$ for all $n \in \mathbb{N}$. It follows from eventual discounting that T^n is a contraction for some $n \in \mathbb{N}$ and hence T is eventually contracting. □

We have an immediate corollary to Lemma A.1 and A.2.

Corollary A.3. *If $v_0 \in bm\mathsf{S}$, the σ-value function v_σ is the unique fixed point of T_σ in $bm\mathsf{S}$ and $T_\sigma^n v \to v_\sigma$ for all $v \in bm\mathsf{S}$. The Bellman operator T has a unique fixed point $\bar{v}$ in $bc\mathsf{S}$ and $T^n w \to \bar{v}$ for all $w \in bc\mathsf{S}$.*

Proof. By Lemma A.1 and a generalized Contraction Mapping Theorem (see, e.g., Cheney, 2013, Section 4.2), T_σ is globally stable on $bm\mathsf{S}$. Hence, if $v_0 \in bm\mathsf{S}$, v_σ is the unique fixed point of T_σ in $bm\mathsf{S}$ and $T_\sigma^n v \to v_\sigma$ for all $v \in bm\mathsf{S}$. The claim for T follows similarly from Lemma A.2. □

Part (b) follows directly from Corollary A.3.

Next we show that $\bar{v}$ given by Corollary A.3 is the value function. First note that $\bar{v} = T\bar{v} \geqslant T_\sigma \bar{v}$ by definition. Iterating T_σ on both sides and using (2), we have $\bar{v} \geqslant T_\sigma^n \bar{v}$. Taking n to infinity, it follows from Corollary A.3 that $\bar{v} \geqslant v_\sigma$. Taking the supremum over Σ gives $\bar{v} \geqslant v^*$.

For the other direction, regularity and the measurable maximum theorem (Aliprantis and Border, 2006, Theorem 18.19) ensure that there exists a $\sigma^* \in \Sigma$ such that $T_{\sigma^*}\bar{v} = T\bar{v}$. Then we have $T_{\sigma^*}\bar{v} = \bar{v}$. Because $\bar{v} \in bc\mathsf{S} \subset bm\mathsf{S}$ and T_{σ^*} has a unique fixed point in $bm\mathsf{S}$ by Corollary A.3, $\bar{v} = v_{\sigma^*}$. By the definition of v^*, we have $v^* \geqslant v_{\sigma^*} = \bar{v}$. Therefore, $v^* = \bar{v}$ and σ^* is the optimal policy. This proves (c) and (d).

One direction of the Bellman's principle of optimality is implied in the argument above. For the other direction, if a policy σ is optimal, then $v_\sigma = v^*$. It follows from Corollary A.3 that $v^* = T_\sigma v^*$. Since $v^* = \bar{v}$ is the fixed point of T, $T_\sigma v^* = Tv^*$. This proves (e).

Part (f) is valid by Corollary A.3 and the fact that $\bar{v} = v^*$.

For part (g), the following proof is adapted from Bertsekas (2013, Proposition 2.4.1).

Let $\{\sigma_k\} \subset \Sigma$ satisfy $T_{\sigma_k} v_{\sigma_{k-1}} = Tv_{\sigma_{k-1}}$. By definition, $T_{\sigma_k} v_{\sigma_{k-1}} = Tv_{\sigma_{k-1}} \geqslant T_{\sigma_{k-1}} v_{\sigma_{k-1}} = v_{\sigma_{k-1}}$. By inequality (2), applying T_{σ_k} to both sides repeatedly gives $T_{\sigma_k}^n v_{\sigma_{k-1}} \geqslant Tv_{\sigma_{k-1}} \geqslant v_{\sigma_{k-1}}$. Taking n to infinity, it follows from Corollary A.3 that $v_{\sigma_k} \geqslant Tv_{\sigma_{k-1}} \geqslant v_{\sigma_{k-1}}$. An inductive argument implies that $v^* \geqslant v_{\sigma_k} \geqslant T^k v_{\sigma_0}$. Taking k to infinity, Corollary A.3 then implies that $v_{\sigma_k} \to v^*$.

A.1.2. *Blackwell's Condition.*

Proposition A.4 (Blackwell's Condition)**.** *Let $\mathcal{D} = (\Gamma, H)$ be a regular dynamic program. If there exists a nonnegative function $\beta \in bc\mathsf{Z}$ and a Feller transition kernel Q on Z such that (β, Q) is eventually discounting and the Bellman operator satisfies*

$$[T(v+c)](x,z) \leqslant (Tv)(x,z) + \beta(z) \int c(z')Q(z,dz') \tag{39}$$

for all $(x,z) \in \mathsf{S}$, $v \in bc\mathsf{S}$, and $c \in bm\mathsf{Z}_+$, then T is eventually contracting on $bc\mathsf{S}$.

Proof of Proposition A.4. For any $v, w \in bc\mathsf{S}$, we have

$$v(x,z) - w(x,z) \leqslant \sup_{x' \in \mathsf{X}} |v(x',z) - w(x',z)| =: c(z)$$

for all $(x,z) \in \mathsf{S}$, where c is lower semicontinuous (Aliprantis and Border, 2006, Lemma 17.29) and thus $c \in bm\mathsf{Z}_+$. Inequality (39) implies that

$$[T(v+c)](x,z) \leqslant (Tv)(x,z) + \beta(z) \int \sup_{x' \in \mathsf{X}} |v(x',z') - w(x',z')| Q(z, dz').$$

It then follows from (2) that

$$(Tv)(x,z) \leqslant (Tw)(x,z) + \beta(z) \int \sup_{x' \in \mathsf{X}} |v(x',z') - w(x',z')| Q(z, dz').$$

Exchanging the roles of v and w, we have

$$|(Tv)(x,z) - (Tw)(x,z)| \leqslant \beta(z) \int \sup_{x' \in \mathsf{X}} |v(x',z') - w(x',z')| Q(z, dz').$$

Iterating on the above inequality, it follows from a similar argument to the proof of Lemma A.2 that $\|T^n v - T^n w\| \leqslant r_n^\beta \|v - w\|$. Since T is a self map on $bc\mathsf{S}$, it follows from eventual discounting that T is eventually contracting. $\square$

A.1.3. *Monotonicity, Concavity, and Differentiability.*

Proof of Theorem 2.2. Since $ibc\mathsf{S}$ is a closed subset of $bc\mathsf{S}$, it suffices to show that T maps $ibc\mathsf{S}$ to functions in $ibc\mathsf{S}$ that are strictly concave in x. For monotonicity, pick any $z \in \mathsf{Z}$ and $v \in ibc\mathsf{S}$. Then for any $y \geqslant x$,

$$\begin{aligned}
(Tv)(y,z) &= H\left(y, z, \sigma^*(y,z), v\right) \\
&\geqslant H\left(y, z, \sigma^*(x,z), v\right) \\
&\geqslant H\left(x, z, \sigma^*(x,z), v\right) = (Tv)(x,z),
\end{aligned}$$

where the first inequality holds because $\sigma^*(x,z) \in \Gamma(x,z) \subset \Gamma(y,z)$ and the second inequality holds because H is increasing in x by Assumption 2.2. For concavity, pick any x, y satisfying $x \neq y$ and $\theta \in (0,1)$ and define $x_\theta = \theta x + (1-\theta) y$. Then, for any $z \in \mathsf{Z}$ and $v \in ibc\mathsf{S}$,

$$\begin{aligned}
\theta(Tv)(x,z) + (1-\theta)(Tv)(y,z) &= \theta H\left(x, z, \sigma^*(x,z), v\right) + (1-\theta) H\left(y, z, \sigma^*(y,z), v\right) \\
&< H\left(x_\theta, z, \theta\sigma^*(x,z) + (1-\theta)\sigma^*(y,z), v\right) \\
&\leqslant H\left(x_\theta, z, \sigma^*(x_\theta, z), v\right) = (Tv)(x_\theta, z),
\end{aligned}$$

where the first inequality holds because $(x, x') \mapsto H(x, z, x', v)$ is strictly concave and the second inequality holds because $\theta\sigma^*(x, z) + (1 - \theta)\sigma^*(y, z) \in \Gamma(x_\theta, z)$ by Assumption 2.2. The strict concavity of H and the maximum theorem imply that $x \mapsto \sigma^*(x, z)$ is single-valued and continuous.

Now we add Assumption 2.3 and consider differentiability. Since $\sigma^*(x_0, z_0) \in \operatorname{int}\Gamma(x_0, z_0)$ and Γ is continuous, there exists an open neighborhood O of x_0 such that $\sigma^*(x_0, z_0) \in \operatorname{int}\Gamma(x, z_0)$ for all $x \in O$. On O we define $W(x) := H(x, z_0, \sigma^*(x_0, z_0), v^*)$. Then $W(x) \leqslant v^*(x, z_0)$ on O and $W(x_0) = v^*(x_0, z_0)$. The claim follows then from Assumption 2.3 and Benveniste and Scheinkman (1979). $\square$

A.2. **Proofs for Section 4.**

Proof of Proposition 4.1. Since $\beta \in bc\mathsf{Z}$, L_β defined in (17) is a bounded linear operator. It follows from Theorem 1.5.5 of Bühler and Salamon (2018) that $r(L_\beta) := \lim_{n\to\infty} \|L_\beta^n\|^{1/n}$ always exists and is bounded above by $\|L_\beta\| = \sup_z \beta(z)$.

Let $\mathbb{1} \equiv 1$ on Z. For each $z \in \mathsf{Z}$ and $n \in \mathbb{N}$, an inductive argument gives

$$\mathbb{E}_z \prod_{t=0}^{n-1} \beta(Z_t) = L_\beta^n \mathbb{1}(z). \tag{40}$$

Thus, eventual discounting can be written as $\|L_\beta^n \mathbb{1}\| < 1$ for some $n \in \mathbb{N}$. Applying Theorem 9.1 of Krasnosel'skii et al. (1972), since (i) L_β is a positive linear operator on $bm\mathsf{Z}$, (ii) the positive cone in this set is solid and normal under the pointwise partial order[27], and (iii) $\mathbb{1}$ lies interior to the positive cone in $bm\mathsf{Z}$, we have

$$r(L_\beta) = \lim_{n\to\infty} \|L_\beta^n \mathbb{1}\|^{1/n} = \lim_{n\to\infty} \left\{ \sup_{z\in\mathsf{Z}} \mathbb{E}_z \prod_{t=0}^{n-1} \beta(Z_t) \right\}^{1/n}, \tag{41}$$

where the second equality is due to (40), nonnegativity of β and the definition of the supremum norm. This confirms the first claim in Proposition 4.1. It also follows immediately that $r(L_\beta) < 1$ implies eventual discounting.

To see that the converse is true, suppose there exists an $n \in \mathbb{N}$ such that $r_n^\beta < 1$. Then any $m \in \mathbb{N}$ can be expressed uniquely as $m = kn + i$ for some $k, i \in \mathbb{N}$ with $i < n$. For

[27] A cone is solid if it has an interior point; it is normal if $0 \leqslant x \leqslant y$ implies that $\|x\| \leqslant M\|y\|$. The cone of nonnegative functions in $bm\mathsf{Z}$ is both solid and normal.

sufficiently large m, it follows from the Markov property that

$$\left(r_m^\beta\right)^{1/m} = \left\{\sup_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{t=0}^{n-1} \beta(Z_t) \left[\mathbb{E}_{Z_{n-1}} \prod_{t=n}^{m-1} \beta(Z_t)\right]\right\}^{1/m}$$
$$\leqslant \left(r_n^\beta r_{m-n}^\beta\right)^{1/m} \leqslant \left(r_n^\beta\right)^{k/m} \left(r_i^\beta\right)^{1/m}.$$

The right hand side is dominated by $(r_n^\beta)^{k/m} M^{1/m}$, where $M := \sup_{i<n} r_i^\beta < \infty$. If $m \to \infty$, then $k/m \to 1/n$, and this term approaches $(r_n^\beta)^{1/n} < 1$. Hence $r(L_\beta) < 1$, as was to be shown. □

Proof of Proposition 4.2. The proof of Proposition 4.1 uses the fact that $r_{\mathsf{B}}(M) = \lim_{n\to\infty} \|M^n h\|_{\mathsf{B}}^{1/n}$ holds when M is a positive (i.e., order preserving) linear operator on a Banach lattice $(\mathsf{B}, \|\cdot\|_{\mathsf{B}})$ with solid positive cone, r_{B} denotes the spectral radius of a linear operator mapping this Banach lattice to itself, and h is interior to the positive cone (Krasnosel'skii et al., 1972, Theorem 9.1). If Z is finite and $\{Z_t\}$ is irreducible with stationary distribution π, we can take B to be all $h\colon \mathsf{Z} \to \mathbb{R}$ and set $\|h\|_{\mathsf{B}} = \sum_{z\in\mathsf{Z}} |h(z)|\pi(z) =: \mathbb{E}_\pi h$. Under this norm, $\mathbb{1}$ is interior to the positive cone of B because, by irreducibility, $\pi(z) > 0$ for all $z \in \mathsf{Z}$. Applying the above expression for the spectral radius to L_β, as well as the result in (40), we obtain

$$r_{\mathsf{B}}(L_\beta) = \lim_{n\to\infty} \|L_\beta^n \mathbb{1}\|_{\mathsf{B}}^{1/n} = \lim_{n\to\infty} \left\{\mathbb{E}_\pi \mathbb{E}_z \prod_{t=0}^{n-1} \beta(Z_t)\right\}^{1/n} = \lim_{n\to\infty} (s_n^\beta)^{1/n}, \tag{42}$$

where the last equality uses the law of iterated expectations and the definition of s_n^β in Proposition 4.2.

It remains only to show that $r_{\mathsf{B}}(L_\beta) = r(L_\beta)$, where the latter is defined, as before, using the supremum norm (see, e.g., (41)). In other words, we need to show that

$$\lim_{n\to\infty} \|L_\beta^n \mathbb{1}\|^{1/n} = \lim_{n\to\infty} \|L_\beta^n \mathbb{1}\|_{\mathsf{B}}^{1/n}. \tag{43}$$

On finite dimensional normed linear spaces, any two norms are equivalent (see, e.g., Bühler and Salamon (2018), Theorem 1.2.5), so we can take positive constants c and d with $\|\cdot\| \leqslant c\|\cdot\|_{\mathsf{B}} \leqslant d\|\cdot\|$ on B. The equality in (43) easily follows and the proof is now complete. □

A.3. **Proofs for Section 5.**

A.3.1. *Homogeneous Functions.* Let the operators T_σ and T be as defined in (4) and (5), respectively, with aggregator H given by (23). The definition of $h_\theta \mathsf{S}$ is given in Section 5.1.

Proof of Proposition 5.1. We first show that T is eventually contracting on $\mathcal{V} = h_\theta \mathsf{S}$. Since Assumption 5.1 holds, the Feller property implies that T maps $\mathcal{V}$ to itself. Note that for any $v \in \mathcal{V}$, we have $v(x, z) = |x|^\theta v(x/|x|, z)$. It follows from Assumption 5.2 that for any $v, w \in \mathcal{V}$,

$$
\begin{aligned}
&|(T^n v)(x_0, z_0) - (T^n w)(x_0, z_0)| \\
&\leqslant \sup_{x_1 \in \Gamma(x_0, z_0)} \beta(z_0) \int \left|(T^{n-1} v)(x_1, z_1) - (T^{n-1} w)(x_1, z_1)\right| Q(z_0, dz_1) \\
&\leqslant \sup_{x_1 \in \Gamma(x_0, z_0)} \beta(z_0) \int |x_1|^\theta \left|(T^{n-1} v)\left(\frac{x_1}{|x_1|}, z_1\right) - (T^{n-1} w)\left(\frac{x_1}{|x_1|}, z_1\right)\right| Q(z_0, dz_1) \\
&\leqslant \sup_{x_1 \in \Gamma(x_0, z_0)} \beta(z_0) \alpha^\theta(z_0) |x_0|^\theta \int \left|(T^{n-1} v)\left(\frac{x_1}{|x_1|}, z_1\right) - (T^{n-1} w)\left(\frac{x_1}{|x_1|}, z_1\right)\right| Q(z_0, dz_1).
\end{aligned}
$$

An inductive argument gives that

$$
\begin{aligned}
&|(T^n v)(x_0, z_0) - (T^n w)(x_0, z_0)| \\
&\leqslant |x_0|^\theta \sup_{x_1 \in \Gamma(x_0, z_0)} \mathbb{E}_{z_0} \prod_{t=0}^{n-1} \beta(z_t) \alpha^\theta(z_t) \left| v\left(\frac{x_n}{|x_n|}, z_n\right) - w\left(\frac{x_n}{|x_n|}, z_n\right)\right| \\
&\leqslant |x_0|^\theta \left(\mathbb{E}_{z_0} \prod_{t=0}^{n-1} \beta(z_t) \alpha^\theta(z_t)\right) \|v - w\|_h
\end{aligned}
$$

where the norm $\|\cdot\|_h$ is defined in (25). Therefore, we have

$$
\|T^n v - T^n w\|_h \leqslant \sup_{z_0 \in \mathsf{Z}} \left(\mathbb{E}_{z_0} \prod_{t=0}^{n-1} \beta(z_t) \alpha^\theta(z_t)\right) \|v - w\|_h.
$$

By Assumption 5.2, T is eventually contracting on $\mathcal{V}$. Hence, T has a unique fixed point $\bar{v}$ on $\mathcal{V}$ and $T^n v \to \bar{v}$ for any $v \in \mathcal{V}$.

Since $T_\sigma v$ is not necessarily in $\mathcal{V}$, we cannot apply the same argument to T_σ. Hence, we prove the remaining results directly. We first show that $v_\sigma := \lim_n (T_\sigma^n \mathbf{0})$ is well defined. It follows from Assumptions 5.1 and 5.2 that

$$\begin{aligned}(T_\sigma^n \mathbf{0})(x_0, z_0) &= \sum_{t=0}^{n-1} \mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i) u(x_t, z_t, \sigma(x_t, z_t)) \\ &\leqslant \sum_{t=0}^{n-1} \mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i) \left| u(x_t, z_t, \sigma(x_t, z_t)) \right| \\ &\leqslant \sum_{t=0}^{n-1} \mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i) \alpha(z_i)^\theta B (1 + \alpha(z_t))^\theta |x_0| \\ &\leqslant \sum_{t=0}^{n-1} \mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i) \alpha(z_i)^\theta B (1 + \bar{\alpha})^\theta |x_0|\end{aligned}$$

where $\bar{\alpha} = \sup_{z \in \mathsf{Z}} \alpha(z)$. It follows from Proposition 4.1 and the Cauchy root test that the series converges absolutely and hence $v_\sigma(x_0, z_0)$ is finite and well defined.

Next we show that $\bar{v} = v^*$. Since $\bar{v} = T\bar{v}$, we have for any $\sigma \in \Sigma$,

$$\begin{aligned}\bar{v}(x_0, z_0) &= \max_{x' \in \Gamma(x_0, z_0)} \left\{ u(x_0, z_0, x') + \beta(z_0) \int \bar{v}(x', z_1) Q(z_0, dz_1) \right\} \\ &\geqslant u(x_0, z_0, \sigma(x_0, z_0)) + \beta(z_0) \int \bar{v}(\sigma(x_0, z_0), z_1) Q(z_0, dz_1).\end{aligned}$$

It follows from induction that

$$\bar{v}(x_0, z_0) \geqslant (T_\sigma^n \mathbf{0})(x_0, z_0) + \mathbb{E}_{z_0} \prod_{t=0}^{n-1} \beta(z_t) \bar{v}(x_n, z_n) \tag{44}$$

where $\{x_n\}$ is given by σ. Since $\bar{v} \in \mathcal{V}$, we have $\bar{v}(x_n, z_n) \leqslant \prod_{t=0}^{n-1} \alpha(z_t)^\theta |x_0|^\theta \|\bar{v}\|_h$. Taking n to infinity in (44), the last term goes to 0 and thus $\bar{v} \geqslant v_\sigma$ for all $\sigma \in \Sigma$. By the measurable maximum theorem, we can find $\sigma^* \in \Sigma$ such that $T\bar{v} = T_{\sigma^*}\bar{v}$. A similar argument shows that σ^* achieves the maximum. Therefore, $\bar{v}$ is the value function and σ^* is the optimal policy.

Because $v^* = T_{\sigma^*} v^*$ is homogeneous of degree θ, we have for any $\lambda \geqslant 0$,

$$v^*(\lambda x, z) = \lambda^\theta v^*(x, z) = \lambda^\theta u(x, z, \sigma^*(x, z)) + \beta(z) \int \lambda^\theta v^*(\sigma^*(x, z), z') Q(z, dz').$$

It follows that $\sigma^*(\lambda x, z) = \lambda \sigma^*(x, z)$, that is, the optimal policy is homogeneous of degree one. $\square$

A.3.2. *Local Contractions.* Recall that the operators T_σ and T are as defined in (4) and (5), respectively, with aggregator H given by (23).

Proof of Proposition 5.2. Define $u_j(x,z) := \max_{x' \in \Gamma(x,z)} |u(x,z,x')|$ if $x \in K_j$ and $r_j := \sup_{x \in K_j, z \in Z} u_j(x,z)$. Since u is continuous and every K_j is compact, $r_j < \infty$ for all j. For any initial state (x_0, z_0), we can find j such that $x_0 \in K_j$. It follows from Assumption 5.3 that $|u(x_t, z_t, x_{t+1})| \leqslant r_j$ for all $t \in \mathbb{N}$.

Choose any increasing and unbounded $\{m_j\}$ such that $m_j \geqslant r_j$. Since Q is Feller, Tv is continuous on every K_j for $v \in c_m\mathsf{S}$, where the space $c_m\mathsf{S}$ is defined in Section 5.2. It follows from Remark 1(a) of Matkowski and Nowak (2011) that $T : c_m\mathsf{S} \to c\mathsf{S}$.

Since $\Gamma(x,z) \subset K_j$ for all $x \in K_j$, we have on K_j

$$\begin{aligned}
|(T^n v)(x,z) - (T^n w)(x,z)| &\leqslant \sup_{x' \in \Gamma(x,z)} \beta(z) \int |T^{n-1}v(x',z') - T^{n-1}w(x',z')| Q(z, dz') \\
&\leqslant \sup_{x' \in K_j} \beta(z) \int |T^{n-1}v(x',z') - T^{n-1}w(x',z')| Q(z, dz') \\
&\leqslant \beta(z) \|T^{n-1}v - T^{n-1}w\|_j.
\end{aligned}$$

An inductive argument gives

$$|(T^n v)(x,z) - (T^n w)(x,z)| \leqslant \mathbb{E}_z \prod_{t=0}^{n-1} \beta(Z_t) \|v - w\|_j.$$

Taking the supremum, we have $\|T^n v - T^n w\|_j \leqslant r_n^\beta \|v - w\|_j$. Since (β, Q) is eventually discounting, T^n is a 0-local contraction for some $n \in \mathbb{N}$.[28] Then it follows from Proposition 1 of Matkowski and Nowak (2011) that T has a unique fixed point $\bar{v}$ in $c_m\mathsf{S}$. It can be proved in the same way that T_σ^n is also a 0-local contraction and hence v_σ is well defined and finite for any initial state. Since we can find σ such that $T_\sigma \bar{v} = T\bar{v}$ by the measurable maximum theorem, the optimality results follow from a similar argument to the proofs of Theorem 2.1. □

A.4. **Proofs for Section 6.**

A.4.1. *Alternative Discount Specifications.* Here we sketch the proof of Theorem 2.1 for the alternative timing when the aggregator satisfies (26). Let $\{Z_t\}$ be a Markov process generated by Q starting at $z = Z_0$ and let $\beta_t = \beta(Z_{t+1})$. A similar argument to the proof of Lemma A.1 yields $|T_\sigma^n v - T_\sigma^n w| \leqslant \mathbb{E}_z \prod_{t=1}^n \beta(Z_{t+1}) \|v - w\|$, where $\mathbb{E}_z$ represents expectation conditional on $Z_0 = z$. Taking the supremum gives $\|T_\sigma^n v - T_\sigma^n w\| \leqslant r_n^\beta \|v - w\|$. Similar result holds for the Bellman operator T. Therefore, both T_σ and

[28] We say an operator $T : c_m\mathsf{S} \to c\mathsf{S}$ is a 0-local contraction if there exists a $\beta \in (0,1)$ such that $\|Tf - Tg\|_j \leqslant \beta \|f - g\|_j$ for all $f, g \in c_m\mathsf{S}$ and all $j \in \mathbb{N}$.

T are eventually contracting if $r_n^\beta < 1$ for some $n \in \mathbb{N}$. The rest of the proof remains the same.

Proof of Proposition 6.1. Recall that the primitives are redefined as in footnote 22. Then the aggregator satisfies

$$|H(x, z, x', v) - H(x, z, x', w)| \leqslant \int \beta(z')|v(x', z') - w(x', z')|\tilde{Q}(z, dz').$$

Based on the discussion above, the eventual discounting condition remains the same. It then follows from Proposition 4.1 that eventual discounting holds if and only if $r(L_\beta) < 1$ and

$$r(L_\beta) = \lim_{n\to\infty} (r_n^\beta)^{1/n} = \lim_{n\to\infty} \left(\sup_{z \in \tilde{\mathsf{Z}}} \tilde{\mathbb{E}}_z \prod_{t=1}^n \beta(\tilde{Z}_{t+1}) \right)^{1/n}$$

where $\tilde{\mathbb{E}}_z$ represents conditional expectation under $\tilde{Q}$. Since $\tilde{Q}$ is induced by Q and $\beta(\tilde{Z}_{t+1}) = bZ_{t+1}/Z_t$, we can write $r_n^\beta = \sup_{z \in \mathsf{Z}} \mathbb{E}_z b^n Z_t$. Then we have $(b^n z_a)^{1/n} \leqslant (r_n^\beta)^{1/n} \leqslant (b^n z_b)^{1/n}$, where z_a and z_b are positive constants such that $z_a < Z_t < z_b$ for all t. Taking $n \to \infty$ gives $r(L_\beta) = b$, so eventual discounting holds if and only if $b < 1$. $\square$

A.4.2. *Epstein-Zin Preferences.* For ease of notation, we replace $1/\psi$ with ρ in what follows. The definition of $\mathcal{V}$ and $\|f\|_I$ are given in Section 6.2. Let the operators T and T_σ be as defined in (4) and (5), respectively, with aggregator H given by (28). Let $\tilde{T}_\sigma$ and $\tilde{T}$ be defined in the same way except that H is replaced by

$$\tilde{H}(x, z, c, v) = \left\{ c^{1-\rho} + \beta(z) \left[\int v\left(R(z)(x - c), z'\right) Q(z, dz') \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}}, \tag{45}$$

which is a special case of H when $\gamma = 0$. We first prove a useful lemma.

Lemma A.5. *$T_\sigma v \leqslant \tilde{T}_\sigma v$ and $Tv \leqslant \tilde{T}v$ for all $v \in \mathcal{V}$.*

Proof. Since $\gamma > 1$, by Jensen's inequality, we have

$$\left[\int v^{1-\gamma}(x, z') Q(z, dz') \right]^{\frac{1}{1-\gamma}} \leqslant \int v(x, z') Q(z, dz')$$

for all $(x, z) \in \mathsf{S}$ and $v \in \mathcal{V}$. It follows that

$$\begin{aligned}(T_\sigma v)(x, z) &\leqslant \left\{ \sigma(x, z)^{1-\rho} + \beta(z) \left[\int v\left[R(z)\left(x - \sigma(x, z)\right), z'\right] Q(z, dz') \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}} \\ &= (\tilde{T}_\sigma v)(x, z).\end{aligned}$$

That $Tv \leqslant \tilde{T}v$ can be shown in a similar way. □

A central result of this section is the following proposition, which guarantees that the σ-value function $v_\sigma = \lim_n (T_\sigma^n \mathbf{0})$ is well defined and a fixed point of T_σ.

Proposition A.6. *Under Assumption 6.1, there exists a function $\hat{v} : \mathsf{S} \to \mathbb{R}_+$ given by*

$$\hat{v}(x_0, z_0) := x_0 \left\{ \lim_{n\to\infty} \sum_{t=0}^{n-1} \left[\mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i)^{\frac{1}{1-\rho}} R(z_i) \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}} \tag{46}$$

such that $\hat{v} \in \mathcal{V}$ and T_σ is a self map on $[0, \hat{v}] \subset \mathcal{V}$. The σ-value function is well defined and is the least fixed point of T_σ on $[0, \hat{v}] \subset \mathcal{V}$. Furthermore, if σ satisfies that $\inf_{z \in \mathsf{Z}} \sigma(x, z)/x > 0$ for all $x > 0$, then v_σ is the unique fixed point of T_σ on $[0, \hat{v}] \subset \mathcal{V}$ and $T_\sigma^n v \to v_\sigma$ for all $v \in [0, \hat{v}] \subset \mathcal{V}$.

We first give two lemmas that are crucial to the proof of Proposition A.6. The first lemma shows that $\hat{v}$ can indeed act as an upper bound function.

Lemma A.7. *$\hat{v} \in \mathcal{V}$ and $T_\sigma \hat{v} \leqslant \hat{v}$ for all $\sigma \in \Sigma$.*

Proof. Let $\hat{v}_n(x_0, z_0) := x_0 A_n(z_0)^{1/(1-\rho)}$ where

$$A_n(z_0) := \sum_{t=0}^{n-1} \left[\mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i)^{\frac{1}{1-\rho}} R(z_i) \right]^{1-\rho}.$$

By Proposition 4.1 and Assumption 6.1, we have

$$\limsup_{n\to\infty} \left[\sup_{z_0 \in \mathsf{Z}} \mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i)^{\frac{1}{1-\rho}} R(z_i) \right]^{\frac{1-\rho}{n}} = r(L_R)^{1-\rho} < 1,$$

where L_R is as defined in (33). It follows from the root test that $\lim_n A_n$ is well defined and bounded on Z. Hence, $\hat{v} = \lim_n \hat{v}_n$ and it satisfies $\|\hat{v}\|_I = \sup_{x \in \mathsf{X}, z \in \mathsf{Z}} |xA(z)/(1 + x)| \leqslant \sup_{z \in \mathsf{Z}} A(z) < \infty$. Therefore, $\hat{v} \in \mathcal{V}$.

Next, we use the operator $\tilde{T}_\sigma$ defined above to show that $T_\sigma \hat{v} \leqslant \hat{v}$. Since A_n is increasing in n, by the Monotone Convergence Theorem, we have $\lim_{n\to\infty} (\tilde{T}_\sigma \hat{v}_n)(x_0, z_0) =$

$(\tilde{T}_\sigma \hat{v})(x_0, z_0)$. Write $A_n(z_0) = \sum_{t=0}^{n-1} B_t(z_0)$. Since $\sigma(x,z) \leqslant x$, it follows that

$$
\begin{aligned}
(\tilde{T}_\sigma \hat{v}_n)(x_0, z_0) &\leqslant x_0 \left\{ 1 + \left[\beta(z_0)^{\frac{1}{1-\rho}} R(z_0) \mathbb{E}_{z_0} A_n(z_1)^{\frac{1}{1-\rho}} \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}} \\
&= x_0 \left\{ 1 + \left[\beta(z_0)^{\frac{1}{1-\rho}} R(z_0) \mathbb{E}_{z_0} \left(\sum_{t=0}^{n-1} B_t(z_1) \right)^{\frac{1}{1-\rho}} \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}}.
\end{aligned}
$$

Since $\rho \in (0,1)$, by the Minkowski inequality, we have

$$
(\tilde{T}_\sigma \hat{v}_n)(x_0, z_0) \leqslant x_0 \left\{ 1 + \sum_{t=0}^{n-1} \left[\beta(z_0)^{\frac{1}{1-\rho}} R(z_0) \mathbb{E}_{z_0} B_t(z_1)^{\frac{1}{1-\rho}} \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}}.
$$

Note that the following equation holds

$$
\beta(z_0)^{\frac{1}{1-\rho}} R(z_0) \mathbb{E}_{z_0} B_t(z_1)^{\frac{1}{1-\rho}} = B_{t+1}(z_0)^{\frac{1}{1-\rho}}
$$

by the Markov property. It follows that

$$
(\tilde{T}_\sigma \hat{v}_n)(x_0, z_0) \leqslant x_0 \left\{ 1 + \sum_{t=1}^{n} B_t(z_0) \right\}^{\frac{1}{1-\rho}} = x_0 A_{n+1}(z_0)^{\frac{1}{1-\rho}} = \hat{v}_{n+1}(x_0, z_0).
$$

Taking n to infinity, we have $\tilde{T}_\sigma \hat{v} \leqslant \hat{v}$. By Lemma A.5, $T_\sigma \hat{v} \leqslant \hat{v}$. □

Lemma A.8. *$T_\sigma v \in \mathcal{V}$ for all $\sigma \in \Sigma$ and $v \in \mathcal{V}$.*

Proof. Evidently $T_\sigma v$ is measurable given $\sigma \in \Sigma$. To see that $T_\sigma v$ is bounded, we have

$$
\begin{aligned}
(T_\sigma v)(x,z) &\leqslant \left\{ \sigma(x,z)^{1-\rho} + \beta(z) \left[\int v \left[R(z) \left(x - \sigma(x,z) \right), z' \right] Q(z, dz') \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}} \\
&\leqslant \left\{ x^{1-\rho} + \beta(z) \|v\|_I^{1-\rho} \left[1 + R(z) x \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}},
\end{aligned}
$$

where the first inequality follows from Lemma A.5 and the second inequality follows from the fact that $\sigma(x,z) \in [0,x]$ and $|v(x,z)| \leqslant \|v\|_I (1+x)$ for all $v \in \mathcal{V}$. Dividing both sides by $(1+x)$ yields (assuming $\sup_z R(z) > 1$)

$$
\|T_\sigma v\|_I \leqslant \sup_{z \in \mathsf{Z}} \left\{ 1 + \beta(z) \|v\|_I^{1-\rho} R(z)^{1-\rho} \right\}^{\frac{1}{1-\rho}}.
$$

Since β and R are bounded, $\|T_\sigma v\|_I < \infty$. □

Proof of Proposition A.6. It is apparent that $T_\sigma \mathbf{0} \geqslant \mathbf{0}$. It follows from Lemma A.7, Lemma A.8, and the monotonicity of T_σ that T_σ is a self map on $[0, \hat{v}] \subset \mathcal{V}$. Let $\{v_n\}$ be a countable chain[29] on $[0, \hat{v}] \subset \mathcal{V}$. Then both $\sup_n v_n$ and $\inf_n v_n$ are measurable

[29] A set $C \subset \mathcal{V}$ is called a chain if for every $x, y \in C$, either $x \leqslant y$ or $y \leqslant x$.

and bounded in norm by $\|\hat{v}\|_I$. So $[0, \hat{v}] \subset \mathcal{V}$ is a countably chain complete partially ordered set. For any increasing $\{v_n\} \subset [0, \hat{v}]$, it follows from the Monotone Convergence Theorem that $\sup_n T_\sigma v_n = T_\sigma(\sup_n v_n)$. Hence, T_σ is monotonically sup-preserving. Then, by the Tarski-Kantrovich Theorem,[30] $v_\sigma := \lim_n (T_\sigma^n \mathbf{0})$ is the least fixed point of T_σ on $[0, \hat{v}] \subset \mathcal{V}$.

If σ satisfies that $\inf_{z \in \mathsf{Z}} (\sigma(x,z)/x) > 0$ for all $x > 0$, then there exists an $\alpha > 0$ such that $\sigma(x,z) \geqslant \alpha x \sup_z A(z) \geqslant \alpha \hat{v}(x,z)$. Since $T_\sigma \mathbf{0} = \sigma \leqslant \hat{v}$, $T_\sigma \mathbf{0}$ and $\hat{v}$ are comparable. Uniqueness and convergence then follow from Theorems 10 and 11 in Marinacci and Montrucchio (2010). □

Recall from Section 6.2 that $\hat{\mathcal{V}}$ is all functions in $\mathcal{V}$ that are homogeneous of degree one in x. The following lemma is useful in the proof of Proposition 6.2.

Lemma A.9. *For any $v \in \hat{\mathcal{V}}$, $Tv \in \hat{\mathcal{V}}$ and there exists a $\sigma \in \Sigma$ homogeneous in x that satisfies $Tv = T_\sigma v$ and $\inf_z \sigma(x,z)/x > 0$ for all $x > 0$.*

Proof. Pick $v \in \hat{\mathcal{V}}$ and we can write $v(x,z) = xh(z)$ for some bounded measurable h. Then (28) becomes

$$H(x,z,c,v) = \left\{ c^{1-\rho} + \beta(z) R(z)^{1-\rho} (x-c)^{1-\rho} \left[\int h(z')^{1-\gamma} Q(z, dz') \right]^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}}. \quad (47)$$

Since $c \mapsto H(x,z,c,v)$ is continuous and $(x,z) \mapsto H(x,z,c,v)$ is measurable, by the measurable maximum theorem, Tv is measurable and there exists a $\sigma \in \Sigma$ such that $T_\sigma v = Tv$. Since $c \leqslant x$ in (47), a similar argument to the proof of Lemma A.8 shows that Tv is bounded in $\| \cdot \|_I$.

In fact, $\sigma(x,z)$ is the solution of the single variable optimization problem maximizing $c^{1-\rho} + (x-c)^{1-\rho} f(z)$ over $0 \leqslant c \leqslant x$ where

$$f(z) := \beta(z) R(z)^{1-\rho} \left[\int h(z')^{1-\gamma} Q(z, dz') \right]^{\frac{1-\rho}{1-\gamma}}.$$

It has closed-form solution $\sigma(x,z) = x/(f(z)^{1/\rho} + 1)$. Therefore, σ is homogeneous in x and thus $Tv = T_\sigma v$ is also homogeneous in x. It follows that $Tv \in \hat{\mathcal{V}}$. Since $f(z)$ is bounded, $\inf_z \sigma(x,z)/x > 0$. □

[30] See, for example, Becker and Rincón-Zapatero (2018) for a version of the theorem and related definitions.

Proof of Proposition 6.2. By Lemma A.9, there exists a σ such that $T_\sigma \hat{v} = T\hat{v}$. It follows from Lemma A.7 that $T_\sigma \hat{v} \leqslant \hat{v}$ and hence $T\hat{v} \leqslant \hat{v}$. Then the monotonicity of T implies that $Tv \leqslant \hat{v}$ for all $v \in \hat{\mathcal{V}}$. By Lemma A.9 and the monotonicity of T, $T^n\mathbf{0}$ is an increasing sequence on $\hat{\mathcal{V}}$ bounded above by $\hat{v}$. Therefore, the pointwise limit $\bar{v} := \lim_{n\to\infty}(T^n\mathbf{0})$ is well defined and is also in $[0, \hat{v}] \subset \hat{\mathcal{V}}$.

To see that $\bar{v}$ is the value function, pick any $\sigma \in \Sigma$. Since $T^n\mathbf{0}$ is an increasing sequence converging to $\bar{v}$, $\bar{v} \geqslant T^n\mathbf{0} \geqslant T^n_\sigma\mathbf{0}$. Taking n to infinity, it follows from Proposition A.6 that $\bar{v} \geqslant v_\sigma$. Next we show that $\bar{v}$ can be achieved by a feasible policy. Since $T^n\mathbf{0} \leqslant \bar{v}$, the monotonicity of T implies that $T^{n+1}\mathbf{0} \leqslant T\bar{v}$. Taking n to infinity yields $\bar{v} \leqslant T\bar{v}$. By Lemma A.9, there exists a homogeneous $\sigma^* \in \Sigma$ that satisfies the interiority condition and $T_{\sigma^*}\bar{v} = T\bar{v}$. Then we have $\bar{v} \leqslant T_{\sigma^*}\bar{v}$ and hence $\bar{v} \leqslant T^n_{\sigma^*}\bar{v}$ by the monotonicity of T_{σ^*}. Taking n to infinity, it follows from Proposition A.6 that $\bar{v} \leqslant v_{\sigma^*}$. Since $\bar{v} \geqslant v_\sigma$ for all $\sigma \in \Sigma$, $\bar{v} = v_{\sigma^*}$. □

For the specification in de Groot et al. (2018) where the lifetime utility satisfies

$$U(C_t, C_{t+1}, \ldots) = \left\{(1-\beta_t)C_t^{1-\rho} + \beta_t \left[\mathbb{E}_t U^{1-\gamma}(C_{t+1}, C_{t+2}, \ldots)\right]^{\frac{1-\rho}{1-\gamma}}\right\}^{\frac{1}{1-\rho}},$$

we can redefine the upper bound function to be

$$\tilde{v}(x_0, z_0) := x_0 \left\{ \lim_{n\to\infty} \sum_{t=0}^{n-1} \left[\mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i)^{\frac{1}{1-\rho}} R(z_i) \left[1-\beta(z_t)\right]^{\frac{1}{1-\rho}} \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}}.$$

Since $\beta(z_t) < 1$, $\tilde{v}$ is bounded above by $\hat{v}$ in (46). Then it can be shown that all the above results hold for the new preference if Assumption 6.1 is satisfied. The proof is omitted.

A.5. **Analytical Expression for the Geometric Mean.** Consider $\beta_t = \exp(\alpha Z_t)$ where $\{Z_t\}$ obeys (20). An inductive argument shows that for all $t \geqslant 1$,

$$Z_t = (1-\rho^t)\mu + \rho^t Z_0 + \sigma_\epsilon(\epsilon_t + \rho\epsilon_{t-1} + \ldots + \rho^{t-1}\epsilon_1). \tag{48}$$

It follows that

$$\sum_{t=0}^{n-1} Z_t = \left(n - \frac{\rho(1-\rho^n)}{1-\rho}\right)\mu + \frac{1-\rho^{n+1}}{1-\rho}Z_0 + \sigma_\epsilon\left(\epsilon_n + \frac{1-\rho^2}{1-\rho}\epsilon_{n-1} + \ldots + \frac{1-\rho^n}{1-\rho}\epsilon_1\right).$$

Exploiting the properties of log-normal distributions, we have

$$\mathbb{E}_z \exp\left(\sum_{t=0}^{n-1} Z_t\right) = \exp\left(n\mu - \frac{\rho(1-\rho^n)}{1-\rho}\mu + \frac{1-\rho^{n+1}}{1-\rho}z + \frac{\sigma_\epsilon^2}{2}\sum_{t=1}^{n} m_t\right)$$

where $m_t = (1-\rho^t)^2/(1-\rho)^2$. Using the law of iterated expectations gives

$$\mathbb{E}\exp\left(\sum_{t=0}^{n-1} Z_t\right) = \exp\left((n+1)\mu + \frac{(1-\rho^{n+1})^2\sigma_\epsilon^2}{2(1-\rho)^2(1-\rho^2)} + \frac{\sigma_\epsilon^2}{2}\sum_{t=1}^{n} m_t\right).$$

Since $m_t \to 1/(1-\rho)^2$, $\sum m_t/n \to 1/(1-\rho)^2$. Therefore,

$$\lim_{n\to\infty}\left(\mathbb{E}\prod_{t=0}^{n-1}\beta_t\right)^{1/n} = \exp\left(\alpha\mu + \frac{\alpha^2\sigma_\epsilon^2}{2(1-\rho)^2}\right). \tag{49}$$

Setting $\alpha = 1$ gives (22). Setting $\mu = \log(b)$ and $\alpha = 1/(1-1/\psi)$ gives (34).

A.6. **Necessity.** In many settings, the eventual discounting condition cannot be weakened without violating finite lifetime values. Here we briefly illustrate this point, using the connection to spectral radii provided in Proposition 4.1.

Consider a standard dynamic program with lifetime rewards $\mathbb{E}\sum_{t\geqslant 0}\beta^t\pi_t$ given constant β and reward flow $\{\pi_t\}$. In this setting, $\beta < 1$ cannot be relaxed without imposing specific conditions on rewards. For example, if there are constants $0 < a \leqslant b$ such that the process $\{\pi_t\}$ satisfies $a \leqslant \pi_t \leqslant b$ for all t, then we clearly have[31]

$$\mathbb{E}\sum_{t\geqslant 0}\beta^t\pi_t < \infty \text{ if and only if } \beta < 1. \tag{50}$$

Eventual discounting has the same distinction once we replace the constant β with a process $\{\beta_t\}$ under standard regularity conditions. For example, if Z is compact and $\beta_t = \beta(Z_t)$ for some $\beta \in bc\mathsf{Z}$ and Q-Markov process $\{Z_t\}$, then

$$\mathbb{E}_z\sum_{t\geqslant 0}\prod_{i=0}^{t-1}\beta_i\,\pi_t < \infty \text{ if and only if } r(L_\beta) < 1. \tag{51}$$

To see this, suppose first that $r(L_\beta) < 1$. Since $\pi_t \leqslant b$, we have

$$\mathbb{E}_z\sum_{t\geqslant 0}\prod_{i=0}^{t-1}\beta_i\,\pi_t \leqslant b\sum_{t\geqslant 0}\mathbb{E}_z\prod_{i=0}^{t-1}\beta_i \leqslant b\sum_{t\geqslant 0}\sup_z\mathbb{E}_z\prod_{i=0}^{t-1}\beta_i = b\sum_{t\geqslant 0}r_t^\beta.$$

By Cauchy's root convergence criterion, the sum $\sum_{t\geqslant 0} r_t^\beta$ will be finite whenever $\limsup_{t\to\infty}(r_t^\beta)^{1/t} < 1$. This holds when $r(L_\beta) < 1$ by Proposition 4.1.

Now suppose instead that $r(L_\beta) \geqslant 1$. By compactness of L_β, positivity of the function β from Assumption 2.1 and the Krein–Rutman Theorem (see, e.g., Theorem 1.2 in Du

[31] The equivalence in (50) is easy to see because, by the Monotone Convergence Theorem, we have $\mathbb{E}\sum_{t\geqslant 0}\beta^t\pi_t = \sum_{t\geqslant 0}\beta^t\mathbb{E}\pi_t$ and, moreover, $0 < a \leqslant \mathbb{E}\pi_t \leqslant b$.

(2006)), there exists a positive function $e \in bc\mathsf{Z}$ such that $L_\beta e = r(L_\beta)e$. Choosing $\gamma > 0$ such that $\gamma e \leqslant 1$, we have

$$\mathbb{E}_z \sum_{t \geqslant 0} \prod_{i=0}^{t-1} \beta_i \, \pi_t \geqslant a\gamma \sum_{t \geqslant 0} L_\beta^t e(z) = a\gamma \sum_{t \geqslant 0} r(L_\beta)^t e(z)$$

when $Z_0 = z$. Since $e > 0$ and $r(L_\beta) \geqslant 1$, the sum diverges to infinity.

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